Part A Differential Equations

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Introduction

From Mods, and maybe from school, you can already solve some differential equations. In this course, we shall learn to solve a much larger variety and also we develop some general theory.

Throughout, we shall use the following convenient abbreviations: we shall write

DEs: for differential equations.

ODEs: for ordinary DEs, i.e. differential equations with only ordinary derivatives.

PDEs: for partial DEs, i.e. differential equations with partial derivatives.

The course contains six topics, with a chapter devoted to each:

The six chapters

1. ODEs and Picard's Theorem (for existence of solutions).
2. Boundary Value Problems (BVPs) for second-order linear ODEs.
3. Plane autonomous systems.
4. First-order quasi-linear PDEs: the method of characteristics.
5. The classification of second-order linear PDEs.
6. Laplace and Fourier transforms (for ODEs and PDEs).

Books

The first choice is P J Collins *Differential and Integral Equations*, O.U.P. (2006), which can be used for the whole course (Chapters 1-7, 14, 15).

Other good books which cover parts of the course include

W E Boyce and R C DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 7th edition, Wiley (2000).

E Kreyszig, *Advanced Engineering Mathematics, 8th Edition*, Wiley (1999).

G F Carrier and C E Pearson, *Partial Differential Equations – Theory and Technique*, Academic (1988).

J Ockendon, S Howison, A Lacey and A Movchan, *Applied Partial Differential Equations*, Oxford (1999) [a more advanced text].

1 ODEs and Picard's Theorem

1.1 What is an ODE?

An ODE is an equation of the form

$$G(x, y, y', y'',\ ...,\ y^{(n)}) = 0$$

regarded as an equation for $y(x)$. Usually this can be solved for the highest derivative and written in the form

$$\frac{d^n y}{dx^x} := y^{(n)}(x) = F(x, y, y',\ ...,\ y^{(n-1)}).$$

Then the *order* of the ODE is n, the order of the highest derivative which appears.
Given an ODE, certain obvious questions arise. We could ask:

- does it have solutions? Can we find them (explicitly or implicitly)? If not, can we at least say something about them?
- given data e.g. the values $y(a)$, $y'(a)$, ... of $y(x)$ and its first $n-1$ derivatives at some initial value a of x, does it have a solution? is it

unique? Are other choices of data possible?

We shall answer the first question in this section, and consider the second in the next section.

For simplicity, we begin with a first-order ODE with data:

$$y' = f(x, y) \text{ with } y(a) = b. \tag{1.1}$$

This is an *initial value problem* or IVP, since we are given y at an initial, or starting, value of x . You know how to solve a variety of equations like this, but you may not have encountered the following difficulty.

1.2 The warning example:

Consider this IVP:

$$y' = 3y^{2/3}; \quad y(0) = 0. \tag{1.2}$$

So separate the variables (a mods technique):

$$\frac{dy}{3y^{2/3}} = dx,$$

integrate and use the data to find

(i) A solution $y = x^3$;

(ii) But evidently there is another solution: by direct checking $y = 0$ will do;

(iii) Now we can see that there are infinitely many. Pick a, b with $a < 0 < b$ and define

$$\begin{aligned} y &= (x-a)^3 \qquad x < a \\ &= 0 \qquad a \leq x < b \\ &= (x-b)^3 \qquad b \leq x \end{aligned}$$

The solution is not unique (in fact far from, it since we've found infinitely many solutions). So for uniqueness in the problem (1.1) we must impose conditions on f. We discover what these are in the process of proving that the solution exists. This will be the first *existence theorem* which you've encountered (mathematicians are unique in proving that certain things exist, and that certain things don't, which is usually easier). The proof is quite technical, certainly the most technical thing in the course. In particular, you need to remember from Mods the *Weierstrass M-test* for convergence of a series of functions. Once we get past the proof of this theorem, most of the rest of the course is methods of solution of particular classes of equation.

To be precise then, we shall seek a solution in a rectangle R about the initial point $(x, y) = (a, b)$, so suppose R is $|x - a| \leq h$, $|y - b| \leq k$ (figure 1.1: all figures are currently at the end).

Figure 1.1: the rectangle R.

Our first assumption is that $f(x, y)$ is continuous in R.

Now integrate (1.1) from a to variable x:

$$[y(x)]_a^x = \int_a^x f(t, y(t))dt = y(x) - y(a)$$

so rearranging

$$y(x) = b + \int_a^x f(t, y(t))dt. \tag{1.3}$$

We have transformed the differential equation to an *integral equation* - the unknown y is given in terms of an integral rather than a differential. There is a general theory of these, but we only need to deal with the particular case of (1.3). The standard approach is to seek a solution by *iteration* or successive approximation.

1.3 Picard's method of successive approximation

We start with an initial guess and then improve it. The guesses, or successive approximations or iterates, are labelled $y_n(x)$ starting with $y_0(x)$. Take

$$\left.\begin{array}{c} y_0(x) = b \\ y_{n+1}(x) = b + \int_a^x f(t, y_n(t))dt \end{array}\right\} \tag{1.4}$$

That is, we start with the simplest guess, that y equals its initial value, and at each stage substitute the current guess into the right-hand-side of (1.3) to get the next guess. We need to know if this process converges, and if it does whether it converges to a solution of the problem (1.3). Consider the differences between successive approximations:

$$\left.\begin{array}{c} e_0(x) = b \\ e_{n+1}(x) = y_{n+1}(x) - y_n(x) \end{array}\right\} \tag{1.5}$$

and note that

$$y_n(x) = \sum_0^n e_k(x). \tag{1.6}$$

We have written $y_n(x)$ as the sum of a series, (1.6) and we need this series to converge, so that we need the differences $e_n(x)$ to get small. With hindsight, this needs some assumptions. The first is easy to state:

P(i): on the rectangle $R = \{(x, y) : \; |x - a| \leq h, \; |y - b| \leq k\}$, the function $f(x, y)$ is continuous; this means its bounded, say by $M \in \mathbb{R}$, so $|f(x, y)| \leq M$ in R; suppose further that the rectangle is constrained by $k \leq Mh$.

Next notice that

$$e_{n+1}(x) = y_{n+1}(x) - y_n(x)$$
$$= \int_a^x [f(t, y_n(t)) - f(t, y_{n-1}(t))]dt$$

and recall that *the integral of the modulus of a function is greater than the modulus of the integral* (because the function can be negative). Therefore

$$|\, e_{n+1}(x)| \leq \left| \int_a^x |f(t, y_n) - f(t, y_{n-1})| \, dt \right|. \tag{1.7}$$

Recall we want the e_n to get smaller. This motivates a definition:

1.4 The Lipschitz condition

A function $f(x, y)$ continuous on a rectangle R satisfies a Lipschitz condition with constant A if $\exists$ real positive A such that

$$|f(x, u) - f(x, v)| \leq A|u - v| \text{ for } (x, u) \in R, \; (x, v) \in R. \tag{1.8}$$

This is a new condition on a function, stronger than being continuous but weaker than being differentiable. It turns out to be the right condition to make Picard's Theorem, which is the existence theorem we want, work.

1.5 Picard's Theorem

The ODE (1.1)

$$y' = f(x, y) \text{ with } y(a) = b.$$

has a solution in the rectangle $R : |x - a| \leq h$, $|y - b| \leq k$ provided:

P(i): f is continuous in R, bounded by M (so $|f(x, y)| \leq M$) and $Mh \leq k$.
P(ii): f satisfies a Lipschitz condition in R.

Furthermore, this solution is unique.

Proof

We want the series in (1.6) to converge as $n \to \infty$. We break the proof into a series of steps:

(i) each $y_n(x)$ is continuous and its graph lies inside R;

continuity follows from (1.4) by induction, continuity of f and properties of the integral; for the graph property, note that

$$|y_{n+1}(x) - b| \leq |\int_a^x |f(t, y_n(t)|dt|$$

$$\leq |M \int_a^x dt| = M|x - a| \leq Mh \leq k$$

where we've used P(i). Thus $y_n(x)$ doesn't get further from b than k, and so the graph of y_n goes right across R (it 'comes out the side not the top or bottom').

Figure 1.2: successive iterates graphed in R.

(ii) The Lipschitz condition P(ii) means that, for some positive A,

$$|f(t, y_n) - f(t, y_{n-1})| \leq A|y_n - y_{n-1}| \quad \text{by (1.8).}$$

(iii) This enables us to prove that the $e_n(x)$ get smaller. From (1.7) and (1.8) for $n \geq 1$:

$$|\, e_{n+1}(x)| \leq |\int_a^x |f(t, y_n) - f(t, y_{n-1})|dt|$$

$$\leq A|\int_a^x |y_n - y_{n-1}|dt|$$

and so

$$|\, e_{n+1}(x)| \leq A|\int_a^x |e_n(t)|dt|. \tag{1.9}$$

Now

$$e_1(x) = y_1(x) - b = \int_a^x f(t, b)dt.$$

By **P(i)**, f is bounded by M so that

$$|e_1(x)| \leq |\int_a^x |f(t, b)|dt| \leq M|x - a|.$$

Using this in (1.9) gives

$$|e_2(x)| \leq \frac{AM}{2}|x - a|^2,$$

and then by induction, using (1.9)

$$|e_n(x)| \leq \frac{A^{n-1}M}{n!}|x - a|^n.$$

This is enough to make the series (1.6) converge:

(iv) By the Weierstrass M-test, the series in (1.6) converges to a continuous function.

This is immediate from (iii), since $|x - a| \leq h$ and the series with n-th term $M\frac{A^{n-1}h^n}{n!}$ converges. Thus

$$y_n(x) \to y_\infty(x)$$

with $y_\infty(x)$ continuous. The iteration has a limit, which is a continuous function. Is this limit a solution of the original problem?

(v) We can take the limit of (1.4) to find

$$y_\infty(x) = b + \int_a^x f(t, y_\infty(t))dt \tag{1.10}$$

(this needs justification: it uses a property of the integral **and** the continuity of f) but the RHS in (1.10) is differentiable; therefore, so is $y_\infty(x)$. Differentiate (1.10) to find

$$y'_\infty(x) = f(x, y_\infty(x))$$

and since also $y_\infty(a) = b$, it's a solution. Are there any more?

(vi) Uniqueness: if $y(x)$ and $Y(x)$ are two solutions of the problem, consider their difference:

$$e(x) = y(x) - Y(x) = \int_a^x (f(t,y) - f(t,Y))dt$$

$$\begin{aligned} \text{so } |\, e(x)| &\leq |\int_a^x |f(t,y) - f(t,Y)|dt| \\ &\leq A|\int_a^x |y - Y|dt| \\ &\leq A|\int_a^x |e(t)|dt| \end{aligned}$$

where we have used the Lipschitz condition. Now $e(x)$ is continuous on R; therefore, it is bounded say $|e(x) \leq B$ so

$$|\, e(x)| \leq A\left|\int_a^x Bdt\right| = AB|x-a|$$

and inductively from this

$$|\, e(x)| \leq BA^n\frac{|x-a|^n}{n!} \leq B\frac{A^n h^n}{n!} \to 0 \text{ as } n \to \infty \text{ and } e(x) = 0.$$

The difference is zero, so the solutions are the same i.e. unique (uniqueness proofs almost always go like this: assume there are two and make their difference vanish).

This ends the proof of Picard's Theorem. Since the warning example doesn't have a unique solution, something goes wrong for it. As an exercise, show that the warning example fails the Lipschitz condition (in any neighbourhood of the initial point).

1.6 Picard for first-order systems

We've worked hard to prove existence and uniqueness for a single first-order ODE (1.1). Now we indicate briefly how the same technique can be used to solve a larger class of problems. Suppose then we have a system of first-order ODEs, (or a *first-order system*) in this case just a pair:

$$y' = f(x,y,z)$$

$$z' = g(x,y,z)$$

with initial data

$$y(a) = b; \; z(a) = c.$$

We introduce a matrix notation (which is then clearly capable of dealing with systems of more than two ODEs):

$$\underline{Y} = \begin{pmatrix} y \\ z \end{pmatrix}, \quad \underline{F} = \begin{pmatrix} f \\ g \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} b \\ c \end{pmatrix};$$

so that the system can be written as

$$\underline{Y}' = \underline{F}(x, \underline{Y}); \;\; \underline{Y}(a) = \underline{B}$$

We have written a system of any size in a way which is formally identical to (1.1). We can set about solving it by iteration, following (1.4) by setting

$$\underline{Y}_{n+1} = \underline{B} + \int_a^x \underline{F}(t, \underline{Y}_n(t))dt.$$

Then this will converge as before if we have a Lipschitz condition

$$||\underline{F}(t, \underline{u}) - \underline{F}(t, \underline{v})|| \leq A||\underline{u} - \underline{v}|| \qquad (1.11)$$

with some definition of "size" for a vector, which could be

$$\left|\left|\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right|\right| = |y_1| + |y_2|.$$

Theorem: Picard for first-order systems

The solution exists and is unique for $\underline{F}$ continuous and Lipschitz in a suitable region

$$|y - b| \leq k_1, \; |z - c| \leq k_2, \; |x - a| \leq h$$

(and this extends to a system of n equations.)

1.7 Picard for Higher Order ODEs

With Picard extended to first-order systems, it is a small step to extend it to a single, higher order ODE. For simplicity, we consider just an IVP for a particular class of second-order ODEs, which are the concern of the next section where we learn to call them *linear*:

$$y'' + p(x)y' + q(x)y = 0$$

with initial data

$$y(a) = b \quad y'(a) = c,$$

and p, q continuous for $|x - a| \leq h$.

To reduce this to a first-order system, introduce $z = y'$ and write

$$y' = z = f(x, y, z)$$

$$z' = -pz - qy = g(x, y, z)$$

with data $y(a) = b, \quad z(a) = c$. This is precisely in the form to which the previous section applies, and its easy to check that (1.11) is satisfied, so we get:

Theorem: Picard for second-order linear ODEs

With the assumptions we've made, the solution exists and is unique.

Clearly this method can be extended to the IVP for an n-th order linear ODE. In particular, this justifies our belief that an n-th order ODE needs n pieces of data to fix a unique solution.

1.8 Things that can go wrong with ODEs

We've proved existence and uniqueness of solution for a variety of situations. What kinds of thing can go wrong?

(i) Non-uniqueness of solution, e.g. if Lipschitz fails. We have the warning example and it is easy to produce other examples.

(ii) Blow up of solutions. Picard may give us a solution for a range of x but the solution may become singular (or more graphically *blow up*) if we try to extend x. As an example, the equation

$$y' = y^2$$

with data $y(0) = b$ has solution $y = \frac{1}{\frac{1}{b} - x}$; so $y \to \infty$ as $x \to \frac{1}{b}$.

For the second example, Picard will give existence in a rectangle but it may be small. E.g. with $b = 1$ and $y' = f(x, y) = y^2$, choose h and k, fixing the rectangle. For the bound on f, $M = (1 + k)^2$ will do, and then to satisfy $Mh \leq k$ we need $h \leq \frac{k}{(1+k)^2} \leq \frac{1}{4}$. Therefore, for the proof of Picard with this initial point, we can't get past $x = \frac{1}{4}$. We could move to a new initial point and start again, but the explicit solution blows up at $x = 1$, so we can't get past that.

1.9 Aside: The Interchange Trick

This is a useful device, helpful with higher order ODEs. Suppose we are interested in the second-order (but not necessarily linear) ODE

$$y'' = f(x, y)$$

with data

$$y(0) = a, \quad y'(0) = b.$$

Integrate once, from 0 to x:

$$y'(x) - b = \int_0^x f(t, y(t))dt$$

or

$$y'(x) = b + \int_0^x f(t, y(t))dt$$

and then integrate again:

$$y(x) - y(0) = \int_0^x bdt + \int_0^x \int_0^t f(s, y(s))dsdt$$

so

$$y(x) = a + bx + \int_0^x \int_0^t f(s, y(s))dsdt.$$

Now in the double integral, interchange the order of integration:

$$\int_0^x \int_0^t f(s, y(s))dsdt = \int_0^x \int_s^x f(s, y(s))dtds.$$

Figure 1.3: the region of integration for the interchange trick (shaded).

Carry out the t-integration:

$$= \int_0^x (x - s)f(s, y(s))ds$$

so that

$$y(x) = a + bx + \int_0^x (x - s)f(s, y(s))ds.$$

This is the interchange trick, and it can be very useful. Here we've written the second-order ODE in terms of an integral equation (replacing two integrations by one) and we could imagine seeking a solution by iteration (though that is beyond this course).

Having interchanged once we could imagine iterating that. Thus as an exercise show that, if

$$y(x) = \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} g(s)ds$$

then

$$\frac{d^n y}{dx^n} = g(x)$$

and

$$y(0) = y'(0) = \ ... \ = y^{(n-1)}(0) = 0.$$

This is the result of interchanging $n-1$ times.

2 Boundary Value Problems for 2nd order linear ODEs

In this chapter, we consider the problem of solving:

$$y''(x) + p(x)y' + q(x)y = f(x), \quad a \le x \le b$$

$$y(a) = A, \ y(b) = B.$$

This is a second-order ODE, so we need two bits of data, but here we are given y at the ends of an interval rather than y and y' at the same (initial) point: this is a *boundary-value problem* (BVP) not an IVP. We are seeking a solution through two specified points in the plane, namely (a, A) and (b, B).

We want first to explain why this equation is *linear*.

A second order linear operator is a map of functions

$$y \to L[y] = P_2(x)y'' + P_1(x)y' + P_0(x)y. \tag{2.1}$$

We usually assume the coefficients $P_i(x)$ are continuous, with $P_2(x) \neq 0$ (i.e. not the zero function). This operator is **linear** in the sense that

$$L[\alpha y_1 + \beta y_2] = \alpha L[y_1] + \beta L[y_2] \tag{2.2}$$

for $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$, but usually we stay with reals) and functions y_1, y_2. Thus its linear in the sense of being a linear transformation on a suitable vector space of differentiable functions.

By **a 2nd-order linear ODE** we mean one of

$$L[y] := P_2y'' + P_1y' + P_0y = 0 : H, \text{ homogeneous}$$
$$L[y] := P_2y'' + P_1y' + P_0y = f : N, \text{ inhomogeneous}$$

for some given f.

2.1 Solutions

The following properties of solutions of H and N are easily established:

(i) the solutions of H form a vector space (since if $L[y_1] = 0 = L[y_2]$ then $L[\alpha y_1 + \beta y_2] = 0$).

(ii) if y_1 and y_2 satisfy N then $y_1 - y_2$ satisfies H, so that the general solution of N may be written

$$y = \underbrace{y_{PI}}_{\text{any solution of } N} + \underbrace{y_{CF}}_{\text{general solution of } H}$$

where y_{PI} is called the *particular integral* and y_{CF} the *complementary function*.

Now we develop more of the vector space language.

2.2 Linear independence of functions

A pair of functions $y_1(x)$, $y_2(x)$ is linearly independent if the only linear combination which vanishes (identically):

$$c_1y_1(x) + c_2y_2(x) = 0$$

has $c_1 = c_2 = 0$. They are linearly *dependent* if such c_i, not both zero, can be found. If they are also differentiable then this would entail

$$c_1y_1' + c_2y_2' = 0,$$

i.e.

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

so that the determinant of the matrix is zero.

Define **the Wronskian** of a pair of function to be this determinant:

$$W(x; y_1, y_2) = y_1 y_2' - y_2 y_1'. \tag{2.3}$$

From what we have just seen:

2.3 Proposition

If two functions are linearly dependent then their Wronskian vanishes.

The converse to this isn't obvious; consider the following pair of (once) differentiable functions:

$$y_1 = \begin{cases} 0 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

$$y_2 = \begin{cases} x^2 & x < 0 \\ 0 & x \geq 0 \end{cases}$$

then $W = 0$. However, if $c_1 y_1 + c_2 y_2 = 0$ then evaluation for positive and negative x shows that $c_1 = c_2 = 0$, so that these functions are in fact linearly independent. To establish a partial converse note the following:

2.4 Proposition

If y_1 and y_2 satisfy

$$H : P_2 y'' + P_1 y' + P_0 y = 0$$

then their Wronskian W satisfies.

$$\frac{W'}{W} = -\frac{P_1}{P_2}.$$

The proof is an easy exercise. Solving for W, we get

$$W = const \times \exp\left[-\int^x \frac{P_1(t)}{P_2(t)} dt \right]. \tag{2.4}$$

In particular, **provided P_2 is nowhere zero**, if $W = 0$ at one point, then $W = 0$ at every point (since in this case the exponential can't vanish so this can only happen if the constant in front of (2.4) is zero).

2.5 A basis of solutions of H

We choose solutions y_1 and y_2 of H with

$$\begin{aligned} y_1(a) = 1, \quad {y_1}'(a) = 0 \\ y_2(a) = 0, \quad {y_2}'(a) = 1. \end{aligned}$$

By the work in Section 1, these exist and are unique at least in a neighbourhood of $x = a$ provided $P_2(a) \neq 0$. Also their Wronskian $W(x)$ has $W(a) = 1$, so is nonzero in this neighbourhood of $x = a$, and they are linearly independent. Do they span the vector space of solutions? Suppose $y(x)$ is any other solution and set

$$Y(x) = y_1(x)y(a) + y_2(x)y'(a).$$

Then this is a solution with

$$Y(a) = y(a);\ Y'(a) = y'(a)$$

and so by uniqueness $Y(x) = y(x)$. Therefore they do span the vector space of solutions, and therefore they are a basis. We can conclude:

2.6 Proposition

(i) The dimension of the space of solutions of H is 2.

(ii) Any pair of solutions of H with $W \neq 0$ is a basis.

Exercise: generalise everything done so far to n-th order linear ODEs.

2.7 Variation of parameters

We now know a good deal about the solutions of H. 'Variation of parameters' is a method to use these to solve N. Recall the distinction:

$$L[y] := P_2y'' + P_1y' + P_0y = \begin{array}{ll} 0 & : H \\ f & : N \end{array},$$

and suppose that H is solved by $y = c_1y_1(x) + c_2y_2(x)$ with linearly independent y_1, y_2. We seek a solution of N of the form

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x), \tag{2.5}$$

i.e. we 'vary the parameters'. Using two functions to find one, we expect to be able to impose another condition on the c_i.
Differentiate (2.5) to find

$$y' = c_1 y_1' + c_2 y_2' + c_1' y_1 + c_2' y_2$$

and now *impose*

$$c_1' y_1 + c_2' y_2 = 0. \tag{2.6}$$

The justification for this is finally that it leads to an explicit formula for the solution of N. Differentiating again,

$$y'' = c_1 y_1'' + c_2 y_2'' + c_1' y_1' + c_2' y_2'$$

so that

$$\begin{aligned} L[y] = P_2(c_1 y_1'' + c_2 y_2'' + c_1' y_1' + c_2' y_2') \\ +P_1(c_1 y_1' + c_2 y_2') \\ +P_0(c_1 y_1 + c_2 y_2). \end{aligned}$$

But, since the y_i satisfy H, this gives N as

$$L[y] = P_2(c_1' y_1' + c_2' y_2') = f. \tag{2.7}$$

Solve (2.6) and (2.7) for c_1' to find

$$c_1' = -\frac{f y_2}{P_2 W}, \;\; W = y_1 y_2' - y_2 y_1'$$

and then

$$c_2' = \frac{f y_1}{P_2 W}.$$

Integrate these to obtain

$$\left. \begin{array}{c} c_1(x) = -\int^x \frac{f(t) y_2(t)}{P_2(t) W(t)} dt \\ c_2(x) = \int^x \frac{f(t) y_1(t)}{P_2(t) W(t)} dt \end{array} \right\} \tag{2.8}$$

The freedom in the choice of lower limit in these integrals gives additive constants in c_i and so, by (2.5), adds a solution of H to this solution of N. We shall see how to exploit this freedom after doing an example.

2.8 An example

Consider the equation (an example of N)

$$y'' + y = \tan x \text{ for } 0 \leq x \leq \frac{\pi}{2}.$$

The corresponding H is

$$y'' + y = 0$$

for which we may choose two linearly-independent solutions as

$$y_1 = \sin x; \; y_2 = \cos x.$$

The Wronskian turns out to be

$$W = y_1 y_2' - y_2 y_1' = -1$$

so by (2.8)

$$c_1 = \int \tan t \cos t \; dt = \alpha_1 - \cos x$$

$$c_2 = \int - \tan t \sin t \; dt = \alpha_2 - \log(\sec x + \tan x) + \sin x$$

where the α_i are arbitrary real constants. Then the solution of N is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= -\underbrace{\cos x \log(\sec x + \tan x)}_{PI} + \underbrace{\alpha_1 \sin x + \alpha_2 \cos x}_{CF}. \end{aligned}$$

2.9 Fixing the boundary values: the Green's function

Suppose we have the BVP

$$P_2 y'' + P_1 y' + P_0 y = f \tag{2.9}$$

with data

$$y(a) = 0 = y(b). \tag{2.10}$$

We may solve this by Variation of Parameters, but choosing y_1 so that $y_1(a) = 0$, and y_2 so that $y_2(b) = 0$. (Assume this can be done; we shall return to this point.) So

$$y(x) = c_1(x) y_1(x) + c_2(x) y_2(x)$$

with the c_i as in (2.8). Then

$$y(a) = c_1(a)y_1(a) + c_2(a)y_2(a)$$

$$y(b) = c_1(b)y_1(b) + c_2(b)y_2(b).$$

We want both these to vanish, and, with the choices made for y_i, they will if we impose $c_2(a) = 0 = c_1(b)$. In this case, instead of (2.8) we have

$$\left.\begin{array}{l} c_1(x) = \int_x^b \frac{f(t)y_2(t)}{P_2(t)W(t)}dt, \\ c_2(x) = \int_a^x \frac{f(t)y_1(t)}{P_2(t)W(t)}dt. \end{array}\right\} \tag{2.11}$$

Thus we fix the c_i completely. Now

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$$

$$= \int_a^x \frac{f(t)y_1(t)y_2(x)}{P_2(t)W(t)}dt + \int_x^b \frac{f(t)y_2(t)y_1(x)}{P_2(t)W(t)}dt$$

which we can write concisely as

$$y(x) = \int_a^b G(x,t)f(t)dt \tag{2.12}$$

where

$$G(x,t) = \left\{ \begin{array}{ll} \frac{y_1(t)y_2(x)}{P_2(t)W(t)} & a \le t \le x \le b \\ \frac{y_2(t)y_1(x)}{P_2(t)W(t)} & a \le x \le t \le b \end{array} \right. . \tag{2.13}$$

We call $G(x,t)$ the *Green's function* for the BVP (2.9)-(2.10).

Note that $G(x,t)$ is defined and continuous on the square $a \le x \le b$, $a \le t \le b$ in the (x,t)-plane, and, as a help to remembering the formula (2.13), observe that it vanishes on the sides of this square.

2.10 An example

Extending the example of section 2.8, consider the BVP

$$y'' + y = f(x) \ \text{ for } 0 \le x \le \frac{\pi}{2}$$

with data

$$y(0) = 0 = y\left(\frac{\pi}{2}\right),$$

and run through the method:

- Identify H, as before, as $y'' + y = 0$.
- Choose solution y_1 with $y_1(0) = 0$ so $y_1 = \sin x$ will do.
- Choose solution y_2 with $y_2\left(\frac{\pi}{2}\right) = 0$ so $y_2 = \cos x$ will do.
- Calculate
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -1.$$
- Note $P_2 = 1$

and so by (2.13)

$$G(x,t) = \begin{cases} -\sin t \cos x & 0 \le t \le x \le \frac{\pi}{2} \\ -\cos t \sin x & 0 \le x \le t \le \frac{\pi}{2} \end{cases}$$

The solution of the BVP is then, by (2.12):

$$y(x) = \int_0^{\frac{\pi}{2}} G(x,t) f(t) dt,$$

with this G.

(One can think of equation (2.12) metaphorically as follows: the BVP is $L[y] = f$ so the solution must be $y = L^{-1}[f]$ and this formula defines L^{-1}.)

2.11 A warning

This procedure will fail if $W = 0$, i.e. if y_1 and y_2 and linearly dependent, i.e. proportional, which means that H has a (nontrivial) solution **vanishing at both ends**. Now think about uniqueness for the BVP: if the solution of N is not unique then the difference between any two gives a solution of H satisfying both boundary conditions, which means vanishing at both ends. We conclude:

- G is well defined iff
- $H+$ boundary conditions has no nontrivial solution iff
- $N+$ boundary conditions has a unique solution.

If any one of these fails, they all do. If that happens, we must revert to Variation of Parameters as in section 2.7.

As an example, consider the problem

$$y'' + y = f(x),\ y(0) = 0 = y(\pi), \tag{2.14}$$

so H is $y'' + y = 0$.

We try to follow the method: choose y_1 so that $y_1(0) = 0$, e.g. $y_1 = \sin x$; but then $y_1(\pi) = 0$ too. H has a solution vanishing at both ends so that G is not defined. We revert to Variation of Parameters, choosing a second, linearly independent, solution y_2 of H, say $y_2 = \cos x$. Then $W = -1$, $P_2 = 1$ and $y = c_1(x)y_1(x) + c_2(x)y_2(x)$ with c_i as in (2.8). Now we try to impose the boundary conditions:

$$y(0) = c_2(0) \text{ therefore } = 0 \text{ and } c_2 = -\int_0^x f(t)\sin t dt$$

by (2.8); but now

$$y(\pi) = -c_2(\pi)$$

and, for the other boundary condition we need this to vanish, which is the condition

$$\int_0^\pi f(t)\sin t dt = 0, \tag{2.15}$$

a necessary condition on f, without which the original problem has no solution.

We conclude that (2.14) has a solution iff f satisfies (2.15). If f does satisfy (2.15), then (2.14) has infinitely many solutions, since $c_1(x)$ is undefined up to an additive constant.

2.12 Example: Infinite limits

Up to now we have considered BVPs on finite intervals. The extension to infinite intervals is straightforward but may need care with the limits at the ends of the interval. Here is an example:

Consider the following BVP problem:

$$u'' - u = f, \text{ for } -\infty < x < \infty$$

where f is bounded and continuous on the whole real line. Show that there is at most one solution which is bounded, and show that it can be written in the form

$$u(x) = \int_{-\infty}^{\infty} K(|x-t|)f(t)dt \tag{2.16}$$

for a K which you should find.

Solution: If u_1 and u_2 are two bounded solutions of the BVP, consider $u = u_1 - u_2$; this is bounded and solves $u'' - u = 0$; the general solution of this homogeneous equation is $u = Ae^x + Be^{-x}$ but this solution can only be bounded for all x if $A = B = 0$. That proves uniqueness.

For the explicit solution, construct the Green's function choosing $y_1 \to 0$ as $x \to -\infty$ and $y_2 \to 0$ as $x \to +\infty$; so $y_1 = e^x$ and $y_2 = e^{-x}$ will do. Then

$$W = -2, \; P_2 = 1$$

so

$$c_1 = -\frac{1}{2}\int_x^{\infty} f(t)e^{-t}dt$$
$$c_2 = -\frac{1}{2}\int_{-\infty}^{x} f(t)e^{t}dt$$

and

$$u = -\frac{1}{2}\int_{-\infty}^{x} e^{t-x}f(t)dt - \frac{1}{2}\int_x^{\infty} e^{x-t}f(t)dt.$$

Define

$$K = -\frac{1}{2}e^{-|x-t|}$$

then this takes the form of (2.16). This is a solution, but is it bounded? Suppose $|f| \leq M$ then

$$\begin{aligned}
|u(x)| &\leq \left|\int_{-\infty}^{\infty} \frac{1}{2}e^{-|x-t|}|f(t)|dt\right| \\
&\leq \frac{M}{2}\int_{-\infty}^{\infty} e^{-|x-t|}dt \\
&\leq M\int_0^{\infty} e^{-t}dt \\
&= M
\end{aligned}$$

so u is bounded, and is therefore the unique bounded solution, by the first part.

3 Plane autonomous systems of ODEs

The definition: a *plane autonomous system of ODEs* is a pair of ODEs of the form;

$$\frac{dx}{dt} = X(x, y) \tag{3.1}$$
$$\frac{dy}{dt} = Y(x, y)$$

Here "autonomous" means there is no t-dependence in X or Y, and "plane" means there are just two equations, so we can draw pictures in the (x, y) -plane, which will then be called the *phase plane*. Given initial values $x(0) = a,\ y(0) = b$, we expect to find a unique solution (this needs X and Y Lipschitz, which we therefore assume) which will define *a trajectory* or *phase path* in the phase plane. It is convenient, though not necessary, to think of t as time, and the trajectory as tracing out the path in time of a moving particle. Then we can put an arrow on the trajectory giving the direction of increasing t.

3.1 Critical points and closed trajectories

A *critical point* is a point in the phase plane where $X = Y = 0$.
A critical point is a particular (very special) trajectory (for which the particle doesn't move). Note that trajectories can only cross at a critical point (since otherwise the particle has more than one velocity there).

There may be trajectories in the phase plane which are closed i.e. which return to the same point. These correspond to **periodic solutions** of (3.1) as may be seen as follows:

suppose that for some finite value p of t, $(x(p), y(p)) = (x(0), y(0))$, while $(x(t), y(t)) \neq (x(0), y(0))$ for $0 < t < p$. Define $\bar{x}(t) = x(t+p)$, $\bar{y}(t) = y(t+p)$. Then

$$\frac{d\bar{x}}{dt} = X(x(t+p),\ y(t+p)) = X(\bar{x}(t),\ \bar{y}(t))$$

and

$$\frac{d\bar{y}}{dt} = Y(\bar{x}(t),\ \bar{y}(t)).$$

So $(\bar{x}(t), \bar{y}(t))$ is another solution with $\bar{x}(0) = x(p) = x(0)$; $\bar{y}(0) = y(p) = y(0)$. Now by uniqueness of solution (given Lipschitz again).

$$x(t+p) = \bar{x}(t) = x(t)$$

$$y(t+p) = \bar{y}(t) = y(t),$$

but this is now true for all t, so a **closed trajectory** corresponds to a **periodic solution** of (3.1) with period p. (It would be possible to have infinite p but then the trajectory has a critical point in its infinite past and the same critical point in its infinite future.)

3.2 An example

Consider the harmonic oscillator equation

$$\ddot{x} = -\omega^2 x.$$

Turn this into a plane autonomous system by introducing y as follows:

$$\left.\begin{array}{rll} & \dot{x} = y & = X(x,y) \\ so & \dot{y} = -\omega^2 x & = Y(x,y). \end{array}\right\} \quad (3.2)$$

(Clearly this trick often works for second-order ODEs arising from Newton's equations.) The only critical point is $(0,0)$, but note that

$$\frac{d}{dt}(\omega^2 x^2 + y^2) = 2\omega^2 x\dot{x} + 2y\dot{y} = 0$$

so $\omega^2 x^2 + y^2 =$ constant. (which, from Mods mechanics, we know to be proportional to the total energy). For a given value of the constant this is the equation of an ellipse, so we can draw all the trajectories in the phase plane as a set of nested (concentric) ellipses:

Figure 3.1: the phase portrait for the harmonic oscillator; to put the arrows on the trajectories, notice that $\dot{x} > 0$ if $y > 0$.

The picture in the phase plane is called the *phase portrait* and from that we see that all trajectories are closed, so all solutions are periodic (as we already know, from Mods).

We want to learn how to sketch the trajectories in the phase plane in general. We start by classifying critical points, so suppose $P = (a,b)$ is a c.p. for (3.1), so

$$X(a,b) = 0 = Y(a,b). \quad (3.3)$$

Now $x = a$, $y = b$ is a solution of (3.1). We analyse its stability by setting

$$x = a + \zeta(t); \; y = b + \eta(t)$$

where ζ and η are thought of as small. From (3.1),

$$\dot{x} = \dot{\zeta} = X(a+\zeta, b+\eta) = X(a,b) + \zeta X_x|_p + \eta X_y|_p + \text{ h.o.}$$

$$y = \dot{\eta} = Y(a,b) + \zeta Y_x|_p + \eta Y_y|_p + \text{ h.o.}$$

where 'h.o.' means quadratic and higher order terms in ζ and η. Now use (3.3) and neglect higher order terms to find

$$\left.\begin{aligned} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}^{\cdot} &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} X_x|_p & X_y|_p \\ Y_x|_p & Y_y|_p \end{pmatrix} \end{aligned}\right\} \tag{3.4}$$

Call this (constant) matrix $\underline{\underline{M}}$ and set $\underline{Z}(t) = \begin{pmatrix} \zeta \\ \eta \end{pmatrix}$ then (3.4) becomes

$$\dot{\underline{Z}} = \underline{\underline{M}}\,\underline{Z}. \tag{3.5}$$

We can solve (3.5) with eigen-vectors and eigen-values as follows: $\underline{Z}_0 e^{\lambda t}$ is a solution, with constant vector $\underline{Z}_0$ and constant scalar λ if

$$\lambda \underline{Z}_0 = \underline{\underline{M}}\ \underline{Z}_0,$$

i.e. $\underline{Z}_0$ is an eigen-vector of $\underline{\underline{M}}$ with eigen-value λ. We are considering just 2×2-matrices, with eigen-values say λ_1 and λ_2 so the general solution if $\lambda_1 \neq \lambda_2$ is

$$\underline{Z}(t) = c_1 \underline{Z}_1 e^{\lambda_1 t} + c_2 \underline{Z}_2 e^{\lambda_2 t}, \tag{3.6}$$

for constant c_i. Recall λ_1, λ_2 may be real, in which case the c_i and the $\underline{Z}_i$ are real, or a complex conjugate pair, in which case the c_i and the $\underline{Z}_i$ are too.

If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ say, we need to take more care. The Cayley-Hamilton Theorem implies that $(\underline{\underline{M}} - \lambda I)^2 = 0$ since the characteristic polynomial is $c_M(x) = (x-\lambda)^2$, so either $M - \lambda I = 0$ or $M - \lambda I \neq 0$. We have a dichotomy:

(i) if $M - \lambda I = 0$ then $M = \lambda I$ and the solution is

$$\underline{Z}(t) = \underline{C} e^{\lambda t}$$

for any constant vector $\underline{C}$.

(ii) if $M - \lambda I \neq 0$ then there exists a constant vector $\underline{Z}_1$ with

$$\underline{Z}_0 := (M - \lambda I)\underline{Z}_1 \neq 0$$

but

$$(M - \lambda I)\underline{Z}_0 = (M - \lambda I)^2 \underline{Z}_1 = 0.$$

One now checks that the solution of (3.5) is

$$(c_1 \underline{Z}_1 + (c_0 + c_1 t)\underline{Z}_0)e^{\lambda t}. \tag{3.7}$$

Now we can use (3.6) and (3.7) to classify critical points.

3.3 Classification of critical points

Case 1. $\lambda_2 > \lambda_1 > 0$ (both real of course)

From (3.6), as $t \to -\infty$, $\underline{Z}(t) \to 0$, while as $t \to +\infty$, $\underline{Z}(t) \sim$ a large multiple of $\underline{Z}_2$, unless $c_2 = 0$ when $\underline{Z}(t) \sim$ a large multiple of $\underline{Z}_1$

Figure 3.2: an unstable node.

These trajectories converge on the critical point into the past, but go off to infinity in the future. A critical point with these properties is called an **unstable node** or an **unstable improper node**.

Case 2: $\lambda_1 < \lambda_2 < 0$ (both real)

This is as above but with $t \to -t$ and the roles of $\underline{Z}_1$, $\underline{Z}_2$ switched. The trajectories converge on the critical point into the future and come in from infinity in the past.

Figure 3.3: a stable node.

This is a **stable node** or **stable improper node**.

Case 3: $\lambda_1 < 0 < \lambda_2$ (both real)

$$\begin{array}{llcr} \text{If } c_1 = 0 \text{ then} & \underline{Z}(t) \to \infty & \text{along } \underline{Z}_2 & \text{as } t \to \infty \\ & \to 0 & \text{as} & t \to -\infty. \\ \text{If } c_2 = 0 \text{ then} & \underline{Z}(t) \to 0 & \text{along } \underline{Z}_1 & \text{as } t \to \infty \\ & \to \infty & & \text{as } t \to -\infty. \end{array}$$

If $c_1, c_2 \neq 0$ then $\underline{Z}(t) \to \infty$ along $\underline{Z}_2$ as $t \to \infty$ along $\underline{Z}_1$ as $t \to -\infty$.

Most trajectories come in approximately parallel to $\pm\underline{Z}_1$ and go out asymptoting to $\pm\underline{Z}_2$.

Figure 3.4: a saddle.

This is a **saddle** (to motivate the name, think of the trajectories as contour lines on a map; then two opposite directions from the critical point are uphill and the two orthogonal directions are downhill).

Case (i) of equal roots is a **proper node**, stable if $\lambda < 0$ and unstable if $\lambda > 0$. Case (ii) of equal roots is again an improper node but the phase portrait is different.

Figure 3.5: unstable proper node case (i) and unstable improper node case (ii)

If the eigen-values are a complex conjugate pair we may write

$$\lambda_1 = \mu - i\nu, \quad \lambda_2 = \mu + i\nu \quad \mu, \nu \in \mathbb{R},$$

and the classification continues in terms of μ and ν.

Case 4: $\mu = 0$

so $\lambda_1 = -i\nu$ and $\lambda_i^2 = -\nu^2 < 0$; in terms of the matrix $\underline{\underline{M}}$ of (3.4), $A+D=0$ but $AD - BC > 0$ so $BC < 0$. Equation (3.4) becomes

$$\begin{pmatrix} \zeta \\ \eta \end{pmatrix}^{\cdot} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}. \tag{3.8}$$

As an exercise, show that now $-C\zeta^2 + 2A\zeta\eta + B\eta^2$ is constant in time. We know that B, C have opposite signs with $(-BC) > A^2$ so this is the equation of an ellipse.

Figure 3.6: a clockwise centre $(B > 0)$.

This case is called a **centre**. The sense of the trajectories, clockwise or anticlockwise, depends on the sign of B; as can be seen from (3.8), $B > 0$ is clockwise (take $\zeta = 0$ and $\eta > 0$, then $\dot{\zeta} = B\eta$ which is positive for positive η i.e. arrow points right).

Case 5: $\mu \neq 0$

So, in (3.6), we must have $\underline{Z}_1 = \bar{\underline{Z}}_2$ and $c_1 = \bar{c}_2$ and

$$\underline{Z}(t) = e^{\mu t}\left[c_1\underline{Z}_1 e^{-i\nu t} + \bar{c}_1\bar{\underline{Z}}_1 e^{i\nu t}\right],$$

which is just like case 4, but with the extra factor $e^{\mu t}$, which is monotonic in time. We have another dichotomy:

(i) $\mu > 0$ then $|\underline{Z}(t)| \to \infty$ as $t \to \infty$ so the trajectory spirals out, into the future. This is called an **unstable spiral**.

Figure 3.7: an unstable spiral; reverse the arrows for a stable spiral

(ii) $\mu < 0$ this is the previous with time reversed so it spirals in, and is called a **stable spiral**.

In case 5, as in case 4, the sense of the spiral is dictated by the sign of B.

Important observation: if $A + D > 0$ then we have one of the cases 1, 3 or 5(i), all of which are **unstable** (but if $A + D < 0$ the critical point can be stable or unstable).

3.4 An example

Find and classify the critical points for the system

$$\dot{x} = x - y = X(x, y)$$

$$\dot{y} = 1 - xy = Y(x, y)$$

solution: for the critical points, from $X = 0$ deduce $x = y$, therefore from $Y = 0$ deduce $x^2 = 1$, and we have either $(1, 1)$ or $(-1, -1)$.

For the classification, calculate

$$M = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -y & -x \end{pmatrix},$$

and evaluate at the cps:

$$\text{at } (1, 1) : M = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} : \ \lambda^2 - 2 = 0 : \ \lambda = \pm\sqrt{2}$$

this is a **saddle**. The corresponding eigenvectors are:

$$\lambda_1 = -\sqrt{2} \quad \underline{Z}_1 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} \text{ direction in}$$

$$\lambda_2 = \sqrt{2} \quad \underline{Z}_2 = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} \text{ direction out}$$

$$\text{at } (-1,-1): \; M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} : \; \lambda^2 - 2\lambda + 2 = 0 : \; \lambda = 1 \pm i.$$

this is an **unstable spiral**; $B < 0$, so its described anticlockwise.

Figure 3.8: the phase portrait.

3.5 Further example: the damped pendulum

Another example from mechanics: a simple plane pendulum with a damping force proportional to the angular velocity. We shall use the analysis of plane autonomous systems to understand the motion.

Take θ to be the angle with the downward vertical, then Newton's equation is

$$ml\ddot{\theta} = -mg\sin\theta - mkl\dot{\theta},$$

where m is the mass of the bob, l is the length of the string, g is the acceleration due to gravity and k is a (real, positive) constant (so that the damping force is mk times the velocity, for simplicity below). We cast this as a plane autonomous system in the usual way: set $x = \theta$ and $y = \dot{x} = \dot{\theta}$ so

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{g}{l}\sin x - ky \end{aligned}$$

For simplicity below, we'll also assume that $k^2 < \frac{4g}{l}$, so that the damping isn't too large.

To sketch the phase portrait, we first find and classify the critical points. The critical points satisfy $y = 0 = \sin x$, so are located at $(x, y) = (N\pi, 0)$. Then

$$M = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x & -k \end{pmatrix}$$

The classification depends on whether N is even or odd:

$$\text{for } x = 2n\pi \quad M = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -k \end{pmatrix}$$

which I claim gives a (clockwise) stable spiral (this needs the condition on k above) ;

$$\text{for } x = (2n+1)\pi \quad M = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -k \end{pmatrix}$$

which I claim gives a saddle.
We now have enough information to sketch the phase portrait (note that $\dot{x}$ is positive or negative according as y is).

Figure 3.9: the phase portrait of the damped pendulum

3.6 An important example: The Lotka–Volterra equations

This is a simplified mathematical model of a *predator-prey* system. Think of variables x standing for the population of prey, and y for the population of predators, both functions of t for time. As time passes, x increases as the prey breed, but decreases as the predators predate; likewise y increases by predation but decreases if too many prey compete. We assume that x and y are governed by the following plane autonomous system:

$$\begin{aligned} \dot{x} &= \alpha x - \gamma x y \\ \dot{y} &= -\beta y + \delta x y, \end{aligned} \tag{3.9}$$

where $\alpha,\ \beta,\ \gamma,\ \delta$ are positive real constants. Because of the interpretation as populations, we only care about $x \geq 0,\ y \geq 0$ but we shall consider the whole plane for simplicity. Again, the aim is to use the analysis of plane autonomous systems to lead us to the phase portrait and an understanding of the dynamics.

For the critical points first, set

$$X := x(\alpha - \gamma y) = 0$$

$$Y := y(-\beta + \delta x) = 0.$$

There are two solutions, $(0,0)$ and $(\frac{\beta}{\delta},\ \frac{\alpha}{\gamma})$. For the matrix:

$$M = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} = \begin{pmatrix} \alpha - \gamma y & -\gamma x \\ \delta y & -\beta + \delta x \end{pmatrix}$$

so first

$$\text{at } (0,0): \ M = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}$$

which gives a saddle, where, it is easy to see, the out-directon is the x-axis and the in-direction is the y-axis. Next

$$\text{at } \left(\frac{\beta}{\delta},\ \frac{\alpha}{\gamma}\right): \ M = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\gamma} & 0 \end{pmatrix} : \lambda^2 + \alpha\beta = 0$$

which gives a centre, described anticlockwise since $B < 0$.

We have found and classified the critical points. Before sketching the phase portrait, it is worth noting, from (3.9), that if $x = 0$ then $\dot{x} = 0$ and if $y = 0$ then $\dot{y} = 0$. Thus the axes are particular trajectories, and trajectories can only cross at critical points (as noted before).

Figure 3.10: the phase portrait for the Lotka-Volterra system

Therefore any trajectory which is ever in the first quadrant is confined to the first quadrant, and no trajectory can enter the first quadrant from outside. Since there is a centre in the first quadrant, it looks as though all trajectories in the first quadrant are **periodic**. This is true, and can be seen by the following argument: form the ratio

$$\frac{\dot{y}}{\dot{x}} = \frac{y(-\beta + \alpha x)}{x(\alpha - \gamma y)} = \frac{dy}{dx}.$$

and separate

$$\frac{(\alpha - \gamma y)}{y} dy - \frac{(-\beta + \delta x)}{x} dx = 0;$$

now integrate

$$\beta \log x - \delta x + \alpha \log y - \gamma y = \text{ const}$$

or

$$x^\beta y^\alpha e^{-\delta x} e^{-\gamma y} = C, \qquad (3.10)$$

for a constant C, which is necessarily positive for a trajectory which is ever in the first quadrant. For different values of C, (3.10) is the equation of the trajectory or equivalently the trajectories are the level sets or contours of the function on the left in (3.10). This function has a single maximum, vanishes on the axes and goes to zero at infinity. Therefore its contours are all closed curves and so all the trajectories are closed and all the solutions of (3.9) are periodic.

This useful technique can be applied to other examples.

3.7 Another important example: limit cycles

Consider the plane autonomous system:

$$\dot{x} = (1 - (x^2 + y^2)^{\frac{1}{2}})x - y$$

$$\dot{y} = (1 - (x^2 + y^2)^{\frac{1}{2}})y + x.$$

Without the first term in each, this is the harmonic oscillator.

Put $x^2 + y^2 = r^2$ then

$$X = x(1 - r) - y$$

$$Y = y(1 - r) + x$$

and one sees that only critical point is $(0, 0)$. One can go through the classification for this, to find that it is an unstable spiral (exercise!) or discover this fact from the following.

We shall transform to polar coordinates. The simplest way to do this is as follows: first

$$r\dot{r} = x\dot{x} + y\dot{y} = x[x(1 - r) - y] + y[y(1 - r) + x]$$

$$= r^2(1 - r)$$

or

$$\dot{r} = r(1 - r).$$

Then, with

$$y = r \sin\theta,$$

we find

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta} = y(1 - r) + x,$$

which gives $\dot{\theta}$, so the system becomes

$$\left.\begin{array}{c} \dot{\theta} = 1 \\ \dot{r} = r(1 - r). \end{array}\right\}$$

Unlike the system in its previous form, we can solve this. First

$$\theta = t + \text{ const},$$

and then

$$dt = \frac{dr}{r(1-r)} = dr\left(\frac{1}{r} + \frac{1}{1-r}\right)$$

so

$$\log\frac{r}{1-r} = t + \text{ const}$$

i.e.

$$\frac{r}{1-r} = Ae^t.$$

Solve for r and change the constant:

$$r = \frac{1}{1+Be^{-t}} = \frac{1}{1+(\frac{1}{r_0}-1)e^{-t}}$$

where $r(0) = r_0$.

Note that as $t \to \infty$, $r \to 1$, while as $t \to -\infty$ either $r \to 0$ if $r_0 < 1$ or $r \to \infty$ at some finite t if $r_0 > 1$.

Now it is clear that the origin is an unstable spiral, and that the trajectories spiral out of it anticlockwise. We can also see that $r = 1$ is a closed trajectory and that all other trajectories (except the fixed point at the origin) tend to it; we call such a closed trajectory a **limit cycle**. It is **stable** because the other trajectories converge on it. (For an example of an unstable limit cycle we could consider the same system but with t changed to $-t$.)

Figure 3.11: phase portrait with a limit cycle

Another system with a limit cycle arises from the *Van der Pol equation*:

$$\ddot{x} + \epsilon(x^2-1)\dot{x} + x = 0$$

where ϵ is a positive real constant. If $\epsilon = 0$ this is the harmonic oscillator again. If $\epsilon \neq 0$ then the usual trick produces a plane autonomous system:

$$\dot{x} = y$$

$$\dot{y} = -\epsilon(x^2-1)y - x.$$

The only critical point is $(0,0)$ and its an unstable spiral for $\epsilon > 0$ (exercise!).

Claim: Its beyond us to show this, but this system has a unique limit cycle, which is stable. There are some good illustrations for this in e.g. Boyce and di Prima (pp 496–500 of the 5th edition).

It's important to be able to detect periodic solution, but can be tricky. We end this section with a discussion of a test that can rule them out.

3.8 The Bendixson–Dulac Theorem

Consider the system $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$. If there exists a function $\varphi(x, y)$ with

$$\rho := \frac{\partial}{\partial x}(\varphi X) + \frac{\partial}{\partial y}(\varphi Y) > 0$$

in a simply connected region R then there can be no (nontrivial) closed trajectories lying entirely in R.

Proof. By nontrivial, I mean I want such a trajectory to have an inside i.e. it isn't just a fixed point. So suppose C is a closed trajectory lying entirely in R; it is nontrivial so there is a disc D lying entirely in R whose boundary is C. Consider the integral

$$\begin{aligned}\int\int_D \rho dx dy &= \int\int_D \left[\frac{\partial}{\partial x}(\varphi X) + \frac{\partial}{\partial y}(\varphi Y)\right] dx dy \\ &= \oint_C -\varphi Y dx + \varphi X dy \\ &= \oint_C -\varphi\left(-\dot{y} dx + \dot{x} dy\right)\end{aligned}$$

but on C, $dx = \dot{x} dt$, $dy = \dot{y} dt$ so this is zero, which contradicts positivity of ρ, so there can be no such C.

3.8.1 Corollary.

If

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$$

has fixed sign in a simply connected region R, then there are no (nontrivial) closed trajectories lying entirely in R.

This is just the previous but with φ const — in an example, always try this first!

3.9 Examples

(i) the damped pendulum (section 3.5)

$$\dot{x} = y$$
$$\dot{y} = -\frac{g}{l}\sin x - ky$$

has no periodic solutions.
Here

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = -k \ < 0;$$

now use the corollary.

(ii)

$$\ddot{x} + f(x)\dot{x} + x = 0$$

has no periodic solutions in a region where f has a fixed sign.

By the usual trick we get the system

$$\dot{x} = y$$
$$\dot{y} = -yf(x) - x$$

then

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = -f(x)$$

and we use the corollary.

(iii) The system

$$\dot{x} = y$$
$$\dot{y} = -x - y + x^2 + y^2$$

has no periodic solutions.

The corollary doesn't help so try the general case:

$$\rho := (\varphi X)_x + (\varphi Y)_y = \varphi(-1+2y) + X\varphi_x + Y\varphi_y.$$

Now guess: $\varphi_y = 0$ then $\rho = \varphi(-1+2y) + y\varphi_x$ so take $\varphi = -e^{-2x}$ and $\rho = 2e^{-2x} > 0$ and we are done.

Clearly, if φ isn't constant we need luck or a hint!

4 First-order quasi-linear PDEs: the method of characteristics

4.1 The problem

In this chapter, we are interested in a PDE of the following form:

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z) \tag{4.1}$$

This is clearly *first-order*, and is *quasi-linear* by being linear in the highest order partial. Note that P, Q are allowed to depend on z, so that neither the whole equation nor even the left-hand-side are actually linear in z. We shall assume that P, Q, R are continuous and Lipschitz in the region of interest, yet to be defined.

There are two problems: find the (or *a*) general solution **or** find a unique solution given data and determine its *domain of definition*. This is the region in the (x,y)-plane in which the solution is uniquely determined by the data. It turns out to depend on both the equation and the data.

The solution of (4.1) will be a function

$$z = f(x,y)$$

but can be thought of as the surface defined by this equation, or equivalently defined by the equation

$$\Sigma(x,y,z) := z - f(x,y) = 0. \tag{4.2}$$

We shall refer to this as the *solution surface* and call it Σ. The method of solution of the equation will be to generate Σ.

Figure 4.1: the solution surface

A normal to the solution surface is defined by

$$\mathbf{n} = \nabla\Sigma = \left(\frac{\partial \Sigma}{\partial x}, \frac{\partial \Sigma}{\partial y}, \frac{\partial \Sigma}{\partial z}\right) = (-f_x, -f_y, 1).$$

(this is a fact from Mods for the single-subject mathematicians; Maths and Comp students should ask their tutors for enlightenment) so consider the vector $\mathbf{t} = (P, Q, R)$. Then

$$\mathbf{t}\cdot\mathbf{n} = -P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R,$$

which vanishes by (4.1), so $\mathbf{t}$ is tangent to the surface Σ.

4.2 The big idea: characteristics

We look for a curve Γ whose tangent is $\mathbf{t}$. If $\Gamma = (x(s), y(s), z(s))$ in terms of a parameter s (*not* necessarily path-length) this means

$$\left.\begin{array}{l} \frac{dx}{ds} = P(x, y, z), \\ \frac{dy}{ds} = Q(x, y, z), \\ \frac{dz}{ds} = R(x, y, z). \end{array}\right\} \tag{4.3}$$

These are the *characteristics equations* and the curve Γ is a *characteristic curve* or just *a characteristic*. Given a characteristic $(x(s), y(s), z(s))$, call the curve $(x(s), y(s), 0)$, which lies below it in the (x, y)-plane, the *characteristic trace* or *characteristic projection*.

The next result shows that characteristics exist, and gives the crucial property of them:

4.3 Proposition

(a) Through each point of space there is a unique characteristic.

(b) Given a point $p \in \Gamma$, the characteristic through p lies entirely on Σ.

Proof

(a) This, with our assumptions on P, Q, R, follows from existence and uniqueness for systems.

(b) (This is fiddly.) Consider the function Σ as a function along Γ; if it stays zero then Γ lies on the surface $\Sigma = 0$. So

$$\Sigma = z(s) - f(x(s), y(s))$$

whence

$$\frac{d\Sigma}{ds} = R(x(s), y(s), z(s)) - \frac{\partial f}{\partial x} P(x(s), y(s), z(s))$$
$$- \frac{\partial f}{\partial y} Q(x(s), y(s), z(s))$$

using (4.3), while

$$R(x,y,f) - \frac{\partial f}{\partial x}P(x,y,f) - \frac{\partial f}{\partial y}Q(x,y,f) = 0.$$

using (4.1). Subtract these

$$\frac{\partial \Sigma}{\partial s} = R(x,y,z) - R(x,y,f) + (P(x,y,z) - P(x,y,f)\frac{\partial f}{\partial x}$$

$$-(Q(x,y,z) - Q(x,y,f))\frac{\partial f}{\partial y}.$$

Now put $z = f + \Sigma$, then the RHS here is a function $F(\Sigma(s), s)$ for some F with $F(0,s) = 0$.

We have an ODE for $\Sigma(s)$, and the RHS is Lifschitz (check!), so it has a unique solution, but $\Sigma(s) = 0$ is a solution, so it's the only one. Therefore, $\Sigma = 0$ all along Γ, so Γ lies on the solution surface. Q.E.D.

Thus the solution surface Σ is *ruled* or *generated* by a collection of characteristics.

4.3.1 Examples of characteristics

We need to gain proficiency in calculating characteristics.

(a) Calculate the characteristics for the PDE:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

From (4.3) write down the characteristic equations and solve them:

$$\frac{dx}{ds} = P = x; \quad x = Ae^s$$

$$\frac{dy}{ds} = Q = y; \quad y = Be^s$$

$$\frac{dz}{ds} = R = z; \quad z = Ce^s$$

with $A,\ B,\ C$ constants (trivial to solve).

(b) Calculate the characteristics for the PDE:

$$(x-z)\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}+z=0.$$

This is a little trickier:

$$\frac{dx}{ds}=x-z; \quad x=Ae^{s}+\frac{C}{2}e^{-s}$$

$$\frac{dy}{ds}=1; \quad y=B+s$$

$$\frac{dz}{ds}=-z; \quad z=Ce^{-s}.$$

with $A,\ B,\ C$ constants. To solve this system, pass over the first, solve the second and third, then come back to the first. (I am adopting a convention to introduce the constants A, B, C in the first, second and third of the characteristic equations respectively.)

Solving the characteristic equations needs experience and luck; there isn't a general algorithm.

4.3.2 The Cauchy problem

Now suppose we are given the solution z of (4.1) along a curve γ_0 (the *data curve*) in the (x, y)-plane. This produces a curve γ in space:

Figure 4.2: geometry of the Cauchy problem.

We introduce a parameter t along γ so it is $(x(t), y(t), z(t))$ while γ_0 is the projection $(x(t), y(t), 0)$. Then, to solve (4.1), we construct the solution surface Σ by taking the characteristics through the points of γ. Thus the method of solution, the *method of characteristics*, is

(i) Parametrise γ as $(x(t), y(t), z(t))$.

(ii) Solve

$$\frac{\partial x}{\partial s}=P$$

$$\frac{\partial y}{\partial s}=Q$$

$$\frac{\partial z}{\partial s}=R$$

for $(x(s,t), y(s,t), z(s,t))$ with data $x(0,t) = x(t);\ y(0,t) = y(t);\ z(0,t) = z(t)$.

then, knowing $(x(s,t), y(s,t), z(s,t))$, we have found Σ *parametrically* i.e. as a function of s and t.

We would like the solution *explicitly*, that is z in terms of x and y, a question to explore below, and there is a restriction on the data for the method to work, also to be found later.

4.4 Examples

(a) Solve

$$(x-z)\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = 0,$$

with $z = 1$ on $x = y$ for $0 < x < \frac{1}{2}$.

Figure 4.3: the data curve for this problem.

We introduce a parameter t for the data, say $\gamma(t) = (t, t, 1)$, for $0 < t < 1/2$, and then solve the characteristic equations (done in section 4.3.1) with this as data at $s = 0$

$$x = Ae^{s} + \frac{C}{2}e^{-s}; \quad x(0,t) = A + \frac{C}{2} = t$$

$$y = B + s; \quad y(0,t) = B = t$$

$$z = Ce^{-s}; \quad z(0,t) = C = 1$$

So, $C = 1$, $B = t$, $A = t - \frac{1}{2}$ and the parametric form of the solution is

$$\left.\begin{array}{l} x = (t-\frac{1}{2})e^{s} + \frac{1}{2}e^{-s} \\ y = t + s \\ z = e^{-s} \end{array}\right\} \qquad (4.4)$$

for $0 < t < \frac{1}{2}$.

(b) Solve

$$z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = -\frac{1}{2}z$$

with $z = \sin x$ on $y = 0$.

So we can take $\gamma(t) = (t, 0, \sin t)$, and solve the characteristic equations:

$$\frac{\partial x}{\partial s} = P = z; \quad x = A - 2Ce^{-\frac{1}{2}s}; \quad A - 2C = t$$

$$\frac{\partial y}{\partial x} = Q = 1; \quad y = B + s; \quad B = 0$$

$$\frac{\partial z}{\partial s} = R = -\frac{1}{2}z; \quad z = Ce^{-\frac{1}{2}s}; \quad C = \sin t$$

where the third column is the value at $s = 0$. So the parametric form of the solution is

$$\left.\begin{array}{l} x = (t + 2\sin t) - 2\sin t\, e^{-\frac{1}{2}s} \\ y = s \\ z = \sin t\, e^{-\frac{1}{2}s} \end{array}\right\} \tag{4.5}$$

4.5 Domain of definition

Where is the solution determined uniquely by the data? This is the domain of definition and will be the region in the (x, y)-plane where the solution surface is a graph of the function $z = f(x, y)$. The solution surface may have edges and may have singularities. We want z as a function of x and y, so we need to be able to eliminate s and t in favour of x and y, at least in principle. For this, recall from Mods the definition of the Jacobian:

$$J = \left|\frac{\partial(x, y)}{\partial(s, t)}\right| = \left|\det\begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix}\right|. \tag{4.6}$$

The significance of J, again from mods, is that it relates the area elements:

$$\mathrm{d}x\ \mathrm{d}y = J\mathrm{d}s\ \mathrm{d}t$$

(this may not be known to Maths and Comp students; ask your tutor for enlightenment). Now a necessary condition to be able to eliminate s and t in favour of x and y is that J be finite and non-zero, because we need non-zero infinitesimal areas on the plane and on the surface to correspond (think about this geometrically); in simply-connected regions of the (x, y)-plane this condition on J is also sufficient - that's harder to prove so we'll just believe it for now.

Let us determine the domain of definition for the example of section 4.4a:

The solution surface has edges given by the characteristics through the ends of γ, which are at $t = 0$ and $t = 1/2$.

At $t = 0$, the characteristic trace is

$$x = -\sinh s; \quad y = s \quad \text{so} \quad x = -\sinh y;$$

at $t = \frac{1}{2}$ it's

$$x = \frac{1}{2}e^{-s} \quad y = s + \frac{1}{2} \quad \text{so} \quad x = \frac{1}{2}e^{\frac{1}{2}-y}$$

Figure 4.4: the domain of definition for this problem.

so the surface lies above the region

$$-\sinh y \leq x \leq \frac{1}{2}e^{\frac{1}{2}-y} \tag{4.7}$$

and these curves don't meet (check!). Next

$$J = \left|\det\begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix}\right|$$

$$= \left|\det\begin{pmatrix} (t-\frac{1}{2})e^s - \frac{1}{2}e^{-s} & 1 \\ e^s & 1 \end{pmatrix}\right|$$

$$= \left|(t - \frac{3}{2})e^s - \frac{1}{2}e^{-s}\right|$$

but $t - \frac{3}{2} < 0$, so this is never zero (or ∞), and the surface has no singularities. Therefore (4.7) gives the domain of definition.

Now look at the example of section 4.4b; there are no edges but

$$J = \left|\det\begin{pmatrix} * & 1 \\ x_t & 0 \end{pmatrix}\right| = |x_t|$$

$$= |1 + 2\cos t(1 - e^{-\frac{1}{2}s})|,$$

where $*$ is unimportant for the determinant.

This is non-zero for $s = 0$ i.e. initially, but it can vanish. Consider the range

$$\log\frac{4}{9} < s < \log 4$$

so

$$\log\frac{3}{2} > -\frac{1}{2}s > -\log 2$$
$$\frac{3}{2} > e^{-\frac{1}{2}s} > \frac{1}{2}$$
$$-1 < 2(1 - e^{-\frac{1}{2}s}) < 1$$

then in this range $1 + 2\cos t(1 - e^{-\frac{1}{2}s}) > 0$. Note $y = s$ here so for

$$-2\log\frac{3}{2} < y < 2\log 2$$

the surface Σ is well-defined, but outside this range Σ "curls up" (J vanishes and the surface becomes vertical before overlapping itself, ceasing to be a graph).

4.6 Cauchy data

We could calculate J at $s = 0$ i.e. at γ. There it is

$$\left|\det\begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix}\right| = \left|\det\begin{pmatrix} P & Q \\ \dot{x} & \dot{y} \end{pmatrix}\right|$$

using a dot for d/dt at $\gamma_0 = (x(t), y(t), 0)$. So on γ_0 we require

$$P(x, y, z)\dot{y} - Q(x, y, z)\dot{x} \neq 0. \tag{4.8}$$

This is a necessary condition on the data for the solution to exist; if it holds, call the data *Cauchy data*. (Geometrically, the condition is that the characteristic traces at γ_0 are *not* tangent to it.)

4.7 Multivalued solutions and blow-up

To get a better grip on what happens when J is 0 or ∞, we shall do another example, which is basic for this phenomenon: solve

$$\left.\begin{array}{l} z_x + zz_y = 0 \\ z = f(y) \text{ on } x = 0. \end{array}\right\} \tag{4.9}$$

We start with the characteristic equations:

$$\frac{\partial x}{\partial s} = P = 1; \quad x = s + A; \quad x = A = 0$$

$$\frac{\partial y}{\partial s} = Q = z; \quad y = B + Cs; \quad y = B = t$$
$$\frac{\partial z}{\partial s} = R = 0; \quad z = C; \quad z = C = f(t)$$

where, as usual, the third column gives the values at $s = 0$. Note that z is constant along any characteristic.

The solution is

$$\left.\begin{array}{l} x = s \\ y = t + sf(t) \\ z = f(t) \end{array}\right\}$$

which we can rearrange as $y - xz = t$; $z = f(t)$ or, by eliminating t,

$$z = f(y - xz).$$

This is an important formula, giving the implicit solution of (4.9), and a lot of information can be obtained from it.

Note that

$$J = \left|\det\begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix}\right| = |1 + sf'(t)|\,.$$

At $s = 0$, $J = 1 \neq 0$, so by (4.8) this *is* Cauchy data, but if $f' < 0$ for any t, then $J = 0$ at $s = \frac{-1}{f'(t)} > 0$. If we think of x as time, so that the data was given at time zero, then a problem has arisen at a finite time after the start.

What is the nature of this problem? Calculate the partial derivative by the chain rule:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial z}{\partial y}(1 + sf') = f'$$

so that

$$\frac{\partial z}{\partial y} = \frac{f'(t)}{1 + sf'(t)}$$

and this goes to ∞ as $s \to -\frac{1}{f'(t)}$, i.e. $\frac{\partial z}{\partial y} \to \infty$ where $J \to 0$. Hence also

$$\frac{\partial z}{\partial x} = -z\frac{\partial z}{\partial y} \to \infty.$$

This is known as a "finite-time blow up" of the partial derivatives (we don't know that z itself is singular).

For more details, we look at a special case: consider the following f

$$f(y) = \begin{array}{cc} 0 & y < 0 \\ -y & 0 \leq y < 1 \\ -1 & y \geq 1 \end{array}$$

We draw the characteristic traces, and recall that z is constant along the characteristic (figure 4.5).

Figure 4.5: the characteristic traces; note the convergence at $x = 1$

From this we can sketch the graph of $z(x, y)$ at $x = 1/2, 1$ and $3/2$ (figure 4.6).

Figure 4.6: the graph of $z(x, y)$ at $x = 1/2, 1$ and $3/2$

From the figures, for $x > 1$, z is *multi-valued*, and the solution is not uniquely determined.

In the context of this example, note that:

1. we could do a smooth version with e.g. $f(y) = -1 - \tanh y$ and get a qualitatively similar picture;

   Figure 4.7: smooth version of figure 4.6 with $f = -1 - \tanh y$.

2. what is happening in the smooth version is that the tangent becomes vertical, so $\frac{\partial z}{\partial y} \to \infty$;

3. if, as happens in some examples, z is the height of a fluid surface, then this is like a breaking wave but if z was e.g. pressure $p(x, y)$ at a point in the plane, we would need a rule to help us choose which of the solutions is the physically correct one. We would then need to learn to live with solutions with discontinuities, interpreting them as "shocks" or "shock waves".

5 Classification of 2nd order linear PDEs

In this section, we are interested in PDEs of the following form:

$$\underbrace{a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy}}_{\text{principal part}} + \underbrace{f(x,y,u,u_x,u_y)}_{\text{linear in } u,u_x,u_y}. \tag{5.1}$$

We know to call these second-order and linear, and we have seen the following examples in Mods:

$$u_{xx} + u_{yy} = 0 \quad \text{Laplace equation}$$

$$u_{xx} - u_{yy} = 0 \quad \text{wave equation if } y = ct$$
$$u_{xx} - u_y = 0 \quad \text{heat equation if } y = t/k.$$

5.1 The idea:

In this section, the key idea is to change coordinates so as to simplify the principal part. Thus

$$(x, y) \to (\varphi(x, y), \psi(x, y));$$

with

$$\frac{\partial(\varphi, \psi)}{\partial(x, y)} = \varphi_x \psi_y - \varphi_y \psi_x \neq 0.$$

For the change in the partials, we calculate

$$u_x = u_\varphi \varphi_x + u_\psi \psi_x; \quad u_y = u_\varphi \varphi_y + u_\psi \psi_y$$

then

$$u_{xx} = u_{\varphi\varphi} \varphi_x^2 + 2u_{\varphi\psi} \varphi_x \psi_x + u_{\psi\psi} \psi_x^2 + u_\varphi \varphi_{xx} + u_\psi \psi_{xx}$$

$$u_{xy} = u_{\varphi\varphi} \varphi_x \varphi_y + u_{\varphi\psi}(\varphi_x \psi_y + \psi_x \varphi_y) + u_{\psi\psi} \psi_x \psi_y + u_\varphi \varphi_{xy} + u_\psi \psi_{xy}$$
$$u_{yy} = u_{\varphi\varphi} \varphi_y^2 + 2u_{\varphi\psi} \varphi_y \psi_y + u_{\psi\psi} \psi_y^2 + u_\varphi \varphi_{yy} + u_\psi \psi_{yy}$$

so that (5.1) becomes

$$A\, u_{\varphi\varphi} + 2B u_{\varphi\psi} + C u_{\psi\psi} + F(\varphi, \psi, u, u_\varphi u_\psi) = 0 \tag{5.2}$$

with

$$\left.\begin{array}{l} A = a\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2 \\ B = a\varphi_x\psi_x + b(\varphi_x\psi_y + \varphi_y\psi_x) + c\varphi_y\psi_y \\ C = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 \end{array}\right\} \tag{5.3}$$

In a matrix notation (5.3) is (check!)

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \varphi_x & \psi_x \\ \varphi_y & \psi_y \end{pmatrix}$$

so that

$$(AC - B^2) = (ac - b^2)(\varphi_x \psi_y - \psi_x \varphi_y)^2. \tag{5.4}$$

(we could obtain (5.4) directly from (5.3) but the matrix notation makes it more perspicuous). Now (5.4) leads to a classification of second-order linear PDEs which is then invariant under change of independent variables:

5.2 The Classification

Second-order linear PDEs are classified into three types as follows:

1. $ac < b^2$ hyperbolic: e.g. wave equation;
2. $ac > b^2$ elliptic: e.g. Laplace equation;
3. $ac = b^2$ parabolic: e.g. heat equation.

We shall look at the classification in terms of the quadratic polynomial

$$a(x,y)\lambda^2 - 2b(x,y)\lambda + c(x,y) = 0. \tag{5.5}$$

Case 1: hyperbolic type

So $ac < b^2$ and the quadratic has distinct real roots λ_1, λ_2.

We solve the first-order quasi-linear PDEs:

$$\varphi_x + \lambda_1(x,y)\varphi_y = 0; \quad \psi_x + \lambda_2(x,y)\psi_y = 0 \tag{5.6}$$

to find new coordinates φ, ψ; then from (5.3) and (5.5) $A = C = 0$; while from (5.4) $B \neq 0$. Divide (5.2) by B to obtain the equation in the form

$$u_{\varphi\psi} + G(\varphi, \psi, u, u_\varphi, u_\psi) = 0. \tag{5.7}$$

This is the *canonical form* for a hyperbolic equation; φ, ψ are *characteristic* or sometimes *canonical variables*; curves on which φ or ψ are constant are *characteristic curves.* We can often solve (5.7) explicitly.

5.3 Examples

(a)

$$u_{xx} - u_{yy} = 0.$$

We already know how to solve this, but let us apply the method. So

$$a = 1,\ b = 0,\ c = -1, \text{and } \lambda^2 - 1 = 0.$$

We can take

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

and solve (5.6)

$$\varphi_x + \varphi_y = 0 \quad \psi_x - \psi_y = 0$$

by

$$\varphi = x - y \quad \psi = x + y.$$

(There is clearly lots of choice at this stage.) The equation has become

$$u_{\varphi\psi} = 0,$$

which we solve at once by

$$u = f(\varphi) + g(\psi),$$

a solution known from Mods.

(b) An example with data: solve

$$xu_{xx} - (x+y)u_{xy} + yu_{yy} + \frac{(x+y)}{(y-x)}(u_x - u_y) = 0$$

with

$$u = \frac{1}{2}(x-1)^2, \quad u_y = 0 \quad \text{on} \quad y = 1.$$

The quadratic (5.5) is

$$x\lambda^2 + (x+y)\lambda + y = 0$$

$$= (\lambda + 1)(x\lambda + y);$$

so choose

$$\lambda_1 = -1 \quad \lambda_2 = -\frac{y}{x}$$

(these roots are real and distinct away from $x = y$, so the equation is hyperbolic except on that line) and solve

$$\varphi_x - \varphi_y = 0; \quad \psi_x - \frac{y}{x}\psi_y = 0$$

by

$$\varphi = x + y; \quad \psi = xy.$$

Calculate

$$u_x = u_\varphi + yu_\psi$$

$$u_y = u_\varphi + xu_\psi$$

so that

$$u_{xx} = u_{\varphi\varphi} + 2yu_{\varphi\psi} + y^2 u_{\psi\psi}$$
$$u_{xy} = u_{\varphi\varphi} + xu_{\varphi\psi} + yu_{\varphi\psi} + xyu_{\psi\psi} + u_\psi$$
$$u_{yy} = u_{\varphi\varphi} + 2xu_{\varphi\psi} + x^2 u_{\psi\psi}$$

(it isn't worth remembering the second-order chain rule, since it is readily obtained from the first-order one when needed.)

Now the PDE becomes

$$\begin{aligned} 0 &= x[u_{\varphi\varphi} + 2yu_{\varphi\psi} + y^2 u_{\psi\psi}] \\ &-(x+y)[u_{\varphi\varphi} + (x+y)u_{\varphi\psi}xyu_{\psi\psi} + u_\psi] \\ &+y[u_{\varphi\varphi} + 2xu_{\varphi\psi} + x^2 u_{\psi\psi}] \\ &+(x+y)u_\psi \\ &= (4xy - (x+y)^2)u_{\varphi\psi} \end{aligned}$$

so

$$u_{\varphi\psi} = 0$$

and the solution is

$$u = f(\varphi) + g(\psi) = f(x+y) + g(xy).$$

To impose the data, calculate

$$u_y = f'(x+y) + xg'(xy)$$

so on $y = 1$,

$$u = f(x+1) + g(x) = \frac{1}{2}(x-1)^2$$
$$u_y = f'(x+1) + xg'(x) = 0.$$

Differentiate the first:

$$f'(x+1) + g'(x) = x - 1$$

and solve simultaneously with the second:

$$g'(x) = -1, \; f'(x+1) = x,$$

so that $f'(x) = x - 1$. Integrate to find

$$g(x) = -x + c_1 \quad f(x) = \frac{x^2}{2} - x + c_2$$

and substitute back in $u(x,1)$:

$$\frac{1}{2}(x-1)^2 = -x + c_1 + \frac{1}{2}(x+1)^2 - (x+1) + c_2$$

to find

$$c_1 + c_2 = 1.$$

Finally

$$u = \frac{1}{2}(x+y)^2 - (x+y) + 1 - xy.$$

Case 2: elliptic type

Now $ac > b^2$ so (5.5) has a complex conjugate pair of roots. Can we solve

$$\varphi_x + \lambda(x,y)\varphi_y = 0 = \psi_x + \bar{\lambda}(x,y)\psi_y?$$

Assume so, with solutions $\psi = \bar{\varphi}$, then $A = C = 0$, $B \neq 0$ and the equation becomes

$$u_{\varphi\bar{\varphi}} + G(\varphi, \bar{\varphi}, u, u_\varphi, u_{\bar{\varphi}}) = 0.$$

Introduce real and imaginary parts for φ as $\varphi = \zeta + i\eta$, $\bar{\varphi} = \zeta - i\eta$ to obtain the canonical form for an elliptic equation:

$$u_{\zeta\zeta} + u_{\eta\eta} + H(\zeta, \eta, u, u_\zeta, u_\eta) = 0, \tag{5.8}$$

which closely resembles the Laplace equation.

Case 3: parabolic type

Now $ac = b^2$ so (5.5) has a repeated root $\lambda(x,y)$. Solve $\varphi_x + \lambda(x,y)\varphi_y = 0$ for one new coordinate, and pick any ψ with $\varphi_x\psi_y - \varphi_y\psi_x \neq 0$ as the other, then $A = B = 0$ so, provided $C \neq 0$, we get the canonical form for a parabolic equation:

$$u_{\psi\psi} + G(u, \varphi, \psi, u_\varphi, u_\psi) = 0,$$

which closely resembles the heat equation.

5.4 Example

Classify and reduce to canonical form the equation

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0. \tag{5.9}$$

The relevant quadratic is

$$x^2\lambda^2 - 2xy\lambda + y^2 = 0 = (x\lambda - y)^2$$

which has equal roots, so this equation is *parabolic*; $\lambda = \frac{y}{x}$ so solve

$$\varphi_x + \frac{y}{x}\varphi_y = 0, \text{ by, e.g., } \varphi = \frac{y}{x}$$

and take, for example, $\psi = x$. Calculate

$$u_x = -\frac{y}{x^2}u_\varphi + u_\psi$$

$$u_y = \frac{1}{x}u_\varphi$$

so that

$$u_{xx} = \frac{y^2}{x^4}u_{\varphi\varphi} - 2\frac{y}{x^2}u_{\varphi\psi} + u_{\psi\psi} + 2\frac{y}{x^3}u_\varphi$$

$$u_{xy} = -\frac{y}{x^3}u_{\varphi\varphi} + \frac{1}{x}u_{\varphi\psi} - \frac{1}{x^2}u_\varphi$$

$$u_{yy} = \frac{1}{x^2}u_{\varphi\varphi}.$$

The equation becomes

$$x^2\left[\frac{y^2}{x^\varphi}u_{\varphi\varphi} + \frac{2y}{x^2}u_{\varphi\psi} + u_{\psi\psi} + \frac{2y}{x^3}u_\varphi\right]$$

$$+2xy\left[-\frac{y}{x^3}u_{\varphi\varphi} + \frac{1}{x}u_{\varphi\psi} - \frac{1}{x^2}u_\varphi\right]$$

$$= y^2\left[\frac{1}{x^2}u_{\varphi\varphi}\right]$$

$$= x^2u_{\psi\psi} = 0$$

so the canonical form is

$$u_{\psi\psi} = 0$$

with general solution $u = F(\varphi) + \psi G(\varphi)$. In terms of the original variables this is:

$$u = F\left(\frac{y}{x}\right) + xG\left(\frac{y}{x}\right). \tag{5.10}$$

NB Very often, a question like this will be phrased in the form 'Classify and reduce to canonical form the equation (5.9) and show that the general solution can be written as (5.10)'. Therefore candidates for φ and ψ are proposed by the question itself.

5.5 A warning example

The type can change e.g. classify the equation

$$u_{xx} + yu_{yy} = 0.$$

Then

$$\lambda^2 + y = 0, \quad \lambda^2 = -y,$$

and this is:

- elliptic in $y > 0$,
- parabolic at $y = 0$,
- hyperbolic in $y < 0$.

5.6 Type and data: well-posed problems

We want to say something about the notion of *well-posed-ness* and its connection with type. Our examples are mostly based on knowledge acquired in Mods.

A problem, consisting of a PDE with data, is said to be *well-posed* if the solution:

- exists
- is unique
- depends *continuously* on the data.

We can't be precise about what *continuous dependence on the data* means, but motivated by the general idea of continuity we take it to mean that a small change in the data (in some sense) leads to a small change in the solution (in possibly some other sense).

We look at some examples from Mods:

(a) The IVP and IBVP (initial-boundary-value problem) for the wave equation

$$u_{xx} - u_{yy} = 0 \quad (ct = y).$$

For the IVP, we know the solution is

$$u = f(x+y) + g(x-y)$$
$$= \frac{1}{2}\left[\varphi(x+y) + \varphi(x-y)\right] + \frac{1}{2}\int_{x-y}^{x+y} \psi(s)ds,$$

where the data are $u(x,0) = \varphi(x)$ and $u_y(x,0) = \psi(x)$. This is d'Alembert's solution of the IVP: it exists, and is unique and, intuitively at least, a small change in φ, ψ gives a small change in u (in fact one can be a bit more precise here). So this problem is well-posed.

For the IBVP consider the set-up:

$$u(x,0) = f(x), \quad u_y(x,0) = g(x) \quad 0 < x < L$$
$$u(0,y) = 0 = u(L,y).$$

So the boundaries are at $x = 0, L$. This IBVP is solved with Fourier series

$$u = \sum_n \sin\frac{n\pi x}{L}\left(a_n \cos\frac{n\pi y}{L} + b_n \sin\frac{n\pi y}{L}\right)$$

with $a_n,\ b_n$ uniquely fixed by $f,\ g$. Again this problem is well-posed (with an appeal to intuition for continuous dependence on the data).

(b) The BVP for the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

Do this first with data at the sides of a square, so $0 \leq x, y \leq a$ with

$$u(0,y) = u(a,y) = u(x,0) = 0;\ u(x,a) = f(x).$$

Consider separable solutions $u_n = \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$, then

$$u = \sum_n a_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

and

$$f(x) = \sum a_n \sinh(n\pi)\ \sin\frac{n\pi x}{a}$$

which determines the solution as a Fourier series.

Now a different BVP, with data at the circumference of the unit circle:

$$\text{on } r = 1,\ u = f(\theta)$$

and in polars

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

The separable solutions are

$$\begin{cases} \left(Ar^n + \frac{B}{r^n}\right)(C\cos n\theta + D\sin n\theta) \\ A + B\log r, \quad n = 0 \end{cases}$$

Regularity at $r = 0$ implies

$$u = \frac{1}{2}a_0 + \sum_1^\infty r^n(a_n \cos n\theta + b_n \sin n\theta)$$

and the boundary value at $r = 1$ requires

$$\frac{1}{2}a_0 + \sum(a_n \cos n\theta + b_n \sin n\theta) = f(\theta)$$

which again is solved by Fourier methods.

For these two cases at least, the BVP for the Laplace equation is well-posed (uniqueness was shown in Mods). It is plausible, but beyond our scope, to show this in general.

(c) The IBVP for the heat equation

$$u_{xx} = u_y$$

on the semi-infinite strip where $y = t > 0$ and $0 < x < L$, and data

$$u(x, 0) = f(x);\ u(0, y) = 0 = u(L, y).$$

The relevant separable solutions are

$$\sin\frac{n\pi x}{L}e^{-\frac{n^2\pi^2 t}{L^2}}$$

so that

$$u = \sum_n a_n \sin\frac{n\pi x}{L}e^{-\frac{n^2\pi^2}{L^2}t}$$

and the initial value requires

$$f(x) = \sum a_n \sin\frac{n\pi x}{L}$$

which is solved by Fourier methods. The solution exists *provided the series for u converges* which it will do for positive t. However, note that for negative t the exponentials grow rapidly with n and there is no reason to expect existence. Uniqueness for this problem was done in Mods, and we can argue plausibly for continuous dependence on data, but we only get well-posed-ness *forward in time.*

(d) What then is not well-posed? We give a few examples:

- BVPs for hyperbolic
  e.g. $u_{xx} - u_{yy} = 0$ on the unit square with data

$$u(0,y) = u(1,y) = u(x,0) = 0;\ u(x,1) = f(x).$$

  Recall this data gave a well-posed problem for the Laplace equation, but here there is no solution at all if $f \neq 0$ (try the Fourier series) while if $f = 0$, then $\sin n\pi x \sin n\pi y$ will do, for any n.

- IBVPs for elliptic
  e.g. $u_{xx} + u_{yy} = 0$ on the semi-infinite strip $0 \leq x \leq 1$, $y \geq 0$, with data

$$u(0,y) = u(1,y) = 0,\ u(x,0) = 1,\ u_y(x,0) = 0.$$

  This data gives a well-posed problem for the wave equation. If we try for separable solutions here, we have $u_n = \sin n\pi x \cosh n\pi y$ so

$$u = \sum_n a_n \sin n\pi x \cosh n\pi y.$$

  Initial conditions need

$$1 = \sum a_n \sin n\pi x$$

  whence

$$\begin{aligned} a_n &= 0 \quad n \text{ even} \\ &= \frac{4}{n^2\pi^2} \quad n \text{ odd}, \end{aligned}$$

  and then

$$u\left(\frac{1}{2}, y\right) = \sum_n \frac{4}{(2n+1)^2\pi^2}(-1)^n \cosh(2n+1)\pi y,$$

  which does not converge for any $y > 0$ (because the cosh terms grow rapidly with n) - there is no solution (strictly speaking, we've only shown that there is no solution of the form considered; we need more).

- the BVP for the heat equation is not well-posed, but we won't show that. It's easier to see that going the wrong direction in time is not well-posed: consider the particular solution

$$u(x,y) = \frac{1}{\sqrt{y}} e^{-\frac{x^2}{4y}}. \tag{5.11}$$

Start with this solution at any positive value of y, say at $y = \epsilon$, then it is analytic in x and its graph looks like the normal distribution; now evolve it towards negative y; at any nonzero value of x it tends to zero as y tends to zero, but at $x = 0$ it diverges, and it makes no sense at all for negative y. The solution has failed to exist within a 'time' ϵ.

Again, it is beyond our scope to prove it in this course, but these different behaviours are universal for the different types of second-order, linear PDEs. In tabulated form, which problems are well-posed?

	IVP	IBVP	BVP
Hyperbolic	yes	yes	no
Elliptic	no	no	yes
Parabolic	yes	yes	no

where the 'yesses' for parabolic are forward in time only.

5.7 Bessel Functions and Legendre Polynomials

We end this chapter with two odd topics on separation of variables, which don't fit anywhere else.

5.7.1 The vibrations of a circular drum in polar coordinates

Suppose the undisturbed drum occupies the disc $r \le a, z = 0$ in plane polar coordinates, and the vertical displacement of the disturbed drum head is given by $z = u(r, \theta, t)$. Then u satisfies the wave equation in the form

$$\frac{1}{c^2} u_{tt} = \nabla^2 u := u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

We seek separable solutions

$$u = R(r)\ \Theta(\theta)\ T(t)$$

with

$$T = \begin{matrix}\cos \\ \sin\end{matrix}\,\omega t$$

so

$$-\frac{\omega^2}{c^2} R\Theta = \Theta(R'' + \frac{1}{r}R') + \frac{R}{r^2}\Theta'',$$

where prime on a function of one variable means derivative with respect to that variable. Divide by $R\Theta$ and separate to find

$$r^2 \left(\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} + \frac{\omega^2}{c^2} \right) = -\frac{\Theta''}{\Theta} \quad \text{therefore} = \text{const.}$$

but we want Θ periodic so the separation constant must be n^2 for integer n:

$$\Theta = \begin{cases} \begin{matrix}\cos \\ \sin\end{matrix}\, n\theta & n \geq 1 \\ 1 & n = 0 \end{cases}$$

$$R^{''} + \frac{1}{r}R^{'} + \left(\frac{\omega^2}{c^2} - \frac{n^2}{r^2} \right) R = 0.$$

Redefine $\rho = \frac{w}{c}r$ to obtain

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\,\frac{dR}{d\rho} + \left(1 - \frac{n^2}{\rho^2} \right) R = 0. \tag{5.12}$$

This (second-order, linear ODE) is *Bessel's equation of order* n. We may seek a series solution of the form

$$R = \rho^\sigma \sum_0^\infty a_k \rho^k$$

in terms of constants σ and a_k, by substituting this into (5.12). This is straightforward to do, but not part of this course. The outcome is that, up to scale (since any constant multiple of the solution has the same property), there is a unique solution which is finite at $\rho = 0$. With a conventional choice of the scale, this solution is called $J_n(\rho)$, *the Bessel function of order* n. The series begins:

$$J_n(\rho) = \frac{\rho^n}{2^n n!} \left[1 - \frac{\rho^2}{4(n+1)} + 0(\rho^4) \right]$$

(it's easy to check by substituting into (5.12) that there is a series solution like this).

Given the series, one can plot the function to obtain graphs like figure 5.1.

Figure 5.1: the first two Bessel functions.

For the problem of the drum, we want $u = 0$ at the rim $r = a$, so $R(a) = 0$ and so $J_n\left(\frac{\omega a}{c}\right) = 0$. Thus $\frac{\omega a}{c} = \alpha_{nm}$, where this is the m-th zero of J_n, which can be found numerically from the series solution. Therefore the allowed frequencies of oscillation of the circular drum are $\omega_{nm} = \frac{c}{a}\alpha_{nm}$. The disturbance u is a combination of terms like

$$J_n\left(\frac{r}{a}\alpha_{nm}\right) \begin{matrix}\cos\\ \sin\end{matrix} n\theta \quad \begin{matrix}\cos\\ \sin\end{matrix} \left(\alpha_{nm}\frac{ct}{a}\right).$$

5.7.2 Laplace's equation in spherical polars

Recall spherical polars are related to Cartesian coordinates by

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta. \end{aligned}$$

and then Laplace's equation becomes

$$\nabla^2 u := u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\sin\theta}(\sin\theta u_\theta)_\theta + \frac{1}{r^2\sin^2\theta}u_{\varphi\varphi} = 0.$$

We could separate this in general as $u(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$, but for simplicity we shall suppose $u_\varphi = 0$, so that $u = R(r)\Theta(\theta)$. Substitute into Laplace's equation:

$$\Theta\left(R'' + \frac{2}{r}R'\right) + \frac{R}{r^2}\,\frac{1}{\sin\theta}(\sin\theta\Theta')' = 0,$$

where prime on a function of one variable again means derivative with respect to that variable. Divide by $R\Theta$ and separate to find

$$\frac{r^2}{R}(R'' + \frac{2}{r}R') = -\frac{1}{\Theta\sin\theta}\,(\sin\theta\Theta')' = \text{ const. } = -\lambda$$

so that

$$R'' + \frac{2}{r}R' + \frac{\lambda}{r^2}R = 0 \tag{5.13}$$
$$(\sin\theta\Theta')' - \lambda\sin\theta\Theta = 0. \tag{5.14}$$

To simplify (5.14), put $\mu = \cos\theta$, so that $\sin\theta\frac{d}{d\theta} = -(1-\mu^2)\frac{d}{d\mu}$ and

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{d\Theta}{d\mu}\right) = \lambda\Theta, \tag{5.15}$$

which is *Legendre's equation*. As with Bessel's equation, we seek series solutions $\Theta = \sum a_k\mu^k$ with constants a_k. This time we find that the series converges for $\mu = \pm 1$ (which it must for solutions to be defined at $\theta = 0, \pi$) if and only if it terminates, i.e. is a polynomial. If the polynomial has degree n, then necessarily $\lambda = -n(n+1)$. These solutions are *Legendre polynomials* P_n and the first few can be taken to be

$$P_0 = 1; \quad P_1 = \mu; \quad P_2 = \frac{1}{2}(3\mu^2 - 1), \quad .... \tag{5.16}$$

For (5.13), this leaves

$$R'' + \frac{2}{r}R' - n\frac{(n+1)}{r^2}R = 0$$

for which solutions are r^n, and r^{-n-1}, so the general solution for u is

$$u = \sum\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta) \tag{5.17}$$

An example:

Consider a BVP for the Laplace equation on the region between concentric spheres of radius a and b, with $0 < a < b$. In spherical polars suppose the solution is $u(r,\theta)$, for the data

$$u(a,\theta) = 1;\; u(b,\theta) = b\cos\theta.$$

The solution is unique by Mods work, and we expect well-posed-ness. From (5.16) and (5.17) we expect a series with just the first two Legendre polynomials P_0 and P_1:

$$u = A_0 + \frac{B_0}{r} + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta,$$

and then the boundary conditions require

$$r = a: \quad A_0 + \frac{B_0}{a} + \left(A_1 a + \frac{B_1}{a^2}\right)\cos\theta = 1,$$

$$r = b \quad A_0 + \frac{B_0}{b} + \left(A_1 b + \frac{B_1}{b^2}\right)\cos\theta = b\cos\theta.$$

For these to hold at all θ, we obtain four equations in four unknowns:

$$\begin{aligned} A_0 + \frac{B_0}{a} &= 1 \\ A_0 + \frac{B_0}{b} &= 0 \\ A_1 a + \frac{B_1}{a^2} &= 0 \\ A_1 b + \frac{B_1}{b^2} &= b \end{aligned}$$

from which a unique solution follows by elementary algebra.

6 Laplace and Fourier Transforms

In this section, we introduce a new idea, the *integral transform* which can convert differential equations into algebraic ones (which are typically easier to solve). We shall consider two such transforms.

6.1 The Laplace Transform

of a function $f(t)$ is

$$\bar{f}(p) = \int_0^\infty f(t)e^{-pt}dt.$$

Note that

- the over-bar does **not** indicate complex conjugate;
- a convenient notation is to write $\bar{f} = \mathcal{L}[f]$;
- we shall often allow p to be complex and then be interested in where $\bar{f}$ is defined in $\mathbb{C}$;

- if $|f(t)| \leq Me^{ct}$ for some positive M, c then the integral is defined at least for some p (in fact for all complex p with $Re(p) > c$); such f is said to have *exponential growth* and examples of such f are any polynomial in t, or any exponential but not e.g. e^{t^2}.

6.2 Examples: a standard list

Check the following standard list of examples of Laplace transforms, with corresponding domain of definition:

	$f(t)$	$\bar{f}(p)$	Domain of definition
1.	1	$\frac{1}{p}$	$Re(p) > 0$
2.	t	$\frac{1}{p^2}$	$Re(p) > 0$
3.	t^n	$\frac{n!}{p^{n+1}}$	$Re(p) > 0$
4.	$e^{at},\ a \in \mathbb{C}$	$\frac{1}{p-a}$	$Re(p) > Re(a)$
5.	$\sin \omega t,\ \omega \in \mathbb{R}$	$\frac{\omega}{p^2+\omega^2}$	$Re(p) > 0$
6.	$\cos \omega t,\ \omega \in \mathbb{R}$	$\frac{p}{p^2+\omega^2}$	$Re(p) > 0$
7.	$\sinh at,\ a \in \mathbb{R}$	$\frac{a}{p^2-a^2}$	$Re(p) > \|a\|$
8.	$\cosh at,\ a \in \mathbb{R}$	$\frac{p}{p^2-a^2}$	$Re(p) > \|a\|$

(to save work, note that (5) – (8) follow from (4).)

6.3 Properties of the Laplace transform

1. Linearity: $\mathcal{L}[\lambda f + \mu g] = \lambda \mathcal{L}[f] + \mu \mathcal{L}[g]$, for $\lambda, \mu \in \mathbb{C}$;

2. If $g(t) = e^{at} f(t)$, then $\bar{g}(p) = \bar{f}(p - a)$;

3. If $g(t) = f'(t)$, then $\bar{g}(p) = p\bar{f}(p) - f(0)$;

4. If $g(t) = \int_0^t f(s)ds$, then $\bar{g}(p) = \frac{\bar{f}(p)}{p}$;

5. Define the *convolution* $f * g$ of f and g by $f * g(t) = \int_0^t f(t-s)g(s)ds$, then $\mathcal{L}[f * g] = \mathcal{L}[f]\mathcal{L}[g]$ or $\overline{f * g} = \overline{f}\ \overline{g}$;

6. If $g(t) = tf(t)$, then $\bar{g}(p) = -\frac{d}{dp}\bar{f}(p)$.

Proofs

1. easy;

2. $\bar{g}(p) = \int_0^\infty g(t)e^{-pt}dt = \int_0^\infty f(t)e^{at}e^{-pt}dt;$

   $= \int_0^\infty f(t)e^{-(p-a)t}dt = \bar{f}(p-a);$

3. $\bar{g}(p) = \int_0^\infty f'(t)e^{-pt}dt = [f(t)e^{-pt}]_0^\infty + p\int_0^\infty f(t)e^{-pt}dt$

   $= -f(0) + p\bar{f}(p);$
4. $\Leftarrow$ (3.)
5. $\mathcal{L}[f * g] = \int_0^\infty \int_0^t f(t-s)g(s)dse^{-pt}dt$

   now interchange the order of integration:
   $= \int_0^\infty \int_s^\infty f(t-s)g(s)e^{-pt}dtds$

   change variable, eliminating t in favour of $u = t - s$
   $= \int_0^\infty \int_0^\infty f(u)g(s)e^{-ps}e^{-pu}duds$

   $= \bar{f}(p)\bar{g}(p) = \mathcal{L}[f]\mathcal{L}[g].$

6. $\frac{d}{dp}\bar{f}(p) = \frac{d}{dp}\int_0^\infty f(t)e^{-pt}dt$

   $= -\int_0^\infty tf(t)e^{-pt}dt = -\bar{g}(p).$

Second derivative: corollary to property 3

If $g(t) = f''(t)$ then $\bar{g}(p) = p^2\bar{f}(p) - pf(0) - f'(0)$.

(Exercise: introduce $h = f'$, so $g = h'$ and use item 3.)

Now let's see the Laplace transform in action:

6.4 Examples

(a) Solve the IVP

$$x'' - 3x' + 2x = 4e^{2t}; \quad x(0) = -3; \quad x'(0) = 5$$

for $x(t)$.

Do $\mathcal{L}$ to the ODE and use property 3 and its corollary:

$$(p^2\bar{x} + 3p - 5) - 3(p\bar{x} + 3) + 2\bar{x} = \frac{4}{p-2}$$

where the RHS comes from our standard list; so

$$\bar{x}(p^2 - 3p + 2) = \frac{4}{p-2} - 3p + 14 = \frac{-3p^2 + 20p - 24}{p-2}$$

which can be solved for $\bar{x}$ as

$$\bar{x} = \frac{-3p^2 + 20p - 24}{(p-1)(p-2)^2}.$$

Split into partial fractions

$$= \frac{-7}{p-1} + \frac{4}{p-2} + \frac{4}{(p-2)^2},$$

and use the standard list to give

$$x = -7e^t + 4e^{2t} + 4te^{2t}$$

where the last term needs $\mathcal{L}[e^{2t}] = \frac{1}{p-2}$, so $\frac{1}{(p-2)^2} = -\frac{d}{dp}\mathcal{L}[e^{2t}] = \mathcal{L}[te^{2t}]$ by property 6.

(b) Solve

$$x'' + \omega^2 x = f(t) \quad x(0) = a, \quad x'(0) = b$$

(this is the equation for a harmonic oscillator with a *driving term* f).

Laplace transform of both sides gives

$$(p^2\bar{x} - ap - b) + \omega^2\bar{x} = \bar{f}$$

so

$$\bar{x}(p^2 + \omega^2) = \bar{f} + ap + b$$

and

$$\bar{x} = \frac{ap}{(p^2+\omega^2)} + \frac{b}{(p^2+\omega^2)} + \frac{\bar{f}}{(p^2+\omega^2)}.$$

The last term, by property 5, is a convolution, so

$$x = a\cos\omega t + \frac{b}{\omega}\sin\omega t + \frac{1}{\omega}f * \sin\omega t$$

or

$$= a\cos\omega t + \frac{b}{\omega}\sin\omega t + \frac{1}{\omega}\int_0^t f(s)\sin\omega(t-s)ds$$

(we could have solved this by the Green's function method, and it's worth comparing the answers.)

(c) an ODE with non-constant coefficients (this doesn't always work).

$$tx'' + 2x' + tx = 0 \quad x(0) = 1,$$

so from the ODE, for a regular solution we also need $x'(0) = 0$.

Laplace transform the equation:

$$\mathcal{L}[tx''] = -\frac{d}{dp}\mathcal{L}[x''] = -\frac{d}{dp}(p^2\bar{x} - p)$$

so

$$-\frac{d}{dp}(p^2\bar{x} - p) + 2(p\bar{x} - 1) - \frac{d}{dp}\bar{x} = 0$$

and

$$-(1+p^2)\frac{d\bar{x}}{dp} - 1 = 0$$

so that

$$-\frac{d\bar{x}}{dp} = \frac{1}{p^2+1} = \mathcal{L}[\sin t]$$

from the standard list. We've reduced a second-order ODE in t to a first-order one in p but we can solve this using property 6, to find $tx = \sin t$ or $x = \frac{1}{t}\sin t$.

In these examples, we have needed some luck to get back from $\bar{x}$ to x, i.e. to invert the Laplace transform. How can this be done in general? We shall postpone this question until after the discussion of the second integral transform.

6.5 The Fourier Transform

Much of the study of this transform can proceed by analogy with the previous. First the definition of the Fourier transform of $f(t)$:

$$f(t) \to \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt.$$

Then note

- existence needs only $\int_{-\infty}^{\infty} |f(t)| dt < \infty$ (this is a fact from the Integration course);
- this is complex from the start, so we may as well take $\omega \in \mathbb{C}$;
- some authors include a factor $\frac{1}{\sqrt{2\pi}}$ in the definition;
- we shall write $\hat{f} = \mathcal{F}[f]$.

6.6 Properties of the Fourier Transform

1. linearity:

$$\mathcal{F}[\lambda f + \mu g] = \lambda \mathcal{F}[f] + \mu \mathcal{F}[g]$$

2. if $g = f'$, then $\hat{g}(\omega) = i\omega \hat{f}(\omega)$ (so no "$f(0)$" as compared to the Laplace transform);

3.

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$$

with the definition $f * g(t) = \int_{-\infty}^{\infty} f(t-s)g(s)ds$ (this definition of convolution agrees with the earlier one if $f = g = 0$ for $t < 0$, but otherwise is different.)

Proofs
Exercises.

6.7 Inversion of the Fourier Tranform

This is given by the *Inversion Formula*:

$$\frac{1}{2}[f(t_-) + f(t_+)] = \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R}^{R} e^{i\omega t} \hat{f}(\omega) d\omega \tag{6.1}$$

where the expression on the left is the average of the limits of f approaching t from above and below. If f is continuous at t this is just $f(t)$.

Proof

This will be just a sketch. I shall indicate where input from Complex Analysis on Integration is required. Recall from Complex Analysis that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2},$$

then

$$\begin{aligned}\int_{-R}^{R} e^{i\omega t} \hat{f}(\omega) d\omega &= \int_{-R}^{R} \int_{-\infty}^{\infty} e^{i\omega t} e^{-i\omega s} f(s) ds d\omega \\ &= \int_{-\infty}^{\infty} \int_{-R}^{R} f(s) e^{i\omega(t-s)} d\omega ds\end{aligned}$$

(OK to change the order of integration?)

$$= \int_{-\infty}^{\infty} \left[\frac{e^{iR(t-s)} - e^{-iR(t-s)}}{i(t-s)} \right] f(s) ds$$

(OK at $t = s$?)

$$\begin{aligned}&= 2 \int_{-\infty}^{\infty} \frac{\sin R(t-s)}{(t-s)} f(s) ds \\ &= 2(I_1 + I_2)\end{aligned}$$

where

$$I_1 = \int_{-\infty}^{t} \frac{\sin R(t-s)}{t-s} f(s) ds = \int_0^\infty \frac{\sin u}{u} f(t - \frac{u}{R}) du$$

with $t - s = \frac{u}{R}$, and

$$I_2 = \int_{t}^{\infty} \frac{\sin R(t-s)}{t-s} f(s) ds = \int_0^\infty \frac{\sin u}{u} f(t + \frac{u}{R}) du$$

with $s-t=\frac{u}{R}$, so

$$\int_{-R}^{R} e^{iwt}\hat{f}(w)dw = 2\int_{0}^{\infty}\frac{\sin u}{u}\left(f(t-\frac{u}{R})+f(t+\frac{u}{R})\right)du.$$

Now take $R\to\infty$ and use the integral with which we began (is the limit OK?). QED.

6.8 An example

Invert

$$\hat{f}=\frac{2}{1+w^2}.$$

The Inversion Formula (6.1) gives

$$f(t)=\frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R}\frac{2e^{i\omega t}}{1+\omega^2}d\omega$$

which we shall evaluate by closing the contour with a semi-circle of radius R centered at the origin. If $t>0$ we choose the semi-circle in the upper-half-plane, and conversely if $t<0$ we take it in the lower-half-plane. Call these contours C_+ and C_- respectively. Poles of the integrand are only at $\pm i$ so for positive t, Cauchy's integral formula gives:

$$\int_{C_+}\frac{e^{i\omega t}}{1+\omega^2}d\omega = 2\pi i\times\frac{1}{2i}e^{-t}=\pi e^{-t},$$

while if $t<0$, we obtain

$$\int_{C_-} = -2\pi i\times(-\frac{1}{2i}e^{t})=\pi e^{t},$$

the extra minus sign coming from the fact that the contour is traversed clockwise. Putting these together we obtain $f(t)=e^{-|t|}$.

6.9 Inversion of Laplace Transform

From (6.1), we can obtain an inversion formula for the Laplace transform as follows:

$$\frac{1}{2}(f(t_-)+f(t_+))=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR}e^{pt}\bar{f}(p)dp \qquad (6.2)$$

where γ is chosen as follows: if $|f(t)| \leq Me^{ct}$ then $\gamma > c$. This means that the line along which the integration is carried out is to the right of any singularities of $\bar{f}$. As before, if f is continuous at t then (6.2) gives $f(t)$.

Proof

This uses Fourier transform and (6.1). Given f, define

$$g(t) = \begin{cases} e^{-\gamma t} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Then

$$\hat{g}(\omega) = \int_0^\infty e^{-i\omega t} e^{-\gamma t} f(t) = \bar{f}(\gamma + i\omega)$$

and, by (6.1)

$$\frac{1}{2}(g(t_- + g(t_+)) = \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R}^{R} e^{i\omega t} \bar{f}(\gamma + i\omega) d\omega$$

so

$$\frac{1}{2}(f(t_-) + f(t_+)) = \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R}^{R} e^{(\gamma + i\omega)t} \bar{f}(\gamma + i\omega) d\omega$$

$$= \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\gamma - iR}^{\gamma + iR} e^{pt} \bar{f}(p) dp \quad \text{where} \quad p = \gamma + i\omega,$$

as required. QED

When it comes to examples, we often evaluate the integral by closing the contour to the left.

6.10 Examples

(a) Invert

$$\bar{f}(p) = \frac{1}{p^2(p-1)}.$$

This example can be done by partial fractions and the standard list, but we shall use the Inversion Formula. Poles of $\bar{f}$ are at 0 and 1 so we need $\gamma > 1$, and we shall close the contour to the left, with an arc of a circle centered at the origin. Write Γ_1 for the straight part of the contour, Γ_2 for the curved arc and Γ for the union. By (6.2), at points of continuity of f,

$$f(t) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\Gamma_1} e^{pt} \frac{dp}{p^2(p-1)}$$

and we claim that, as $R \to \infty$

$$\int_{\Gamma_2} e^{pt} \frac{dp}{p^2(p-1)} \to 0,$$

so consider

$$\oint_{\Gamma} e^{pt} \frac{dp}{p^2(p-1)} = 2\pi i \times \text{ sum of residues.}$$

Calculate

$$Res|_1 = e^t; \quad Res|_0 = -(1+t),$$

and take the limit to find that $f = -(1+t) + e^t$.

(b) Invert $\bar{f}(p) = \frac{1}{p^{1/2}}$.

This has a branch point at the origin. We proceed as before, closing the contour to the left with a circular arc centered at the origin, but we need a 'key-hole', excluding the negative real axis and the origin.

Figure 6.1: the contour for inverting $p^{-1/2}$.

Then we may define $p^{1/2} = r^{1/2}e^{i\theta/2}$ for $-\pi < \theta < \pi$ and consider the integral $\oint_\Gamma e^{pt}\frac{dp}{p^{1/2}}$. With points as labelled on the diagram and DEF a circle of radius ϵ, we claim that

$$\oint_\Gamma = 0; \text{ while } \int_B^C, \quad \int_G^A, \quad \int_{DEF} \to 0$$

$$\int_C^D e^{pt} \frac{dp}{p^{1/2}} \to \int_0^\infty e^{-rt} \frac{dr}{ir^{1/2}}$$

$$\int_F^G e^{pt} \frac{dp}{p^{1/2}} \to \int_0^\infty e^{-rt} \frac{dr}{ir^{1/2}}$$

so

$$\int_A^B \to -\frac{2}{i} \int_0^\infty \frac{e^{-rt}}{r^{1/2}} dr$$

and

$$f(t) = \frac{1}{\pi}\int_0^\infty e^{-rt}\frac{dr}{r^{1/2}}.$$

To evaluate this integral, put $rt = s^2$,

$$= \frac{1}{\pi}\int_0^\infty e^{-s^2}\,\frac{2sds}{t}\,\frac{t^{1/2}}{s}$$

$$= \frac{1}{\pi\sqrt{t}}\int_{-\infty}^\infty e^{-s^2}ds$$

which is a standard integral:

$$= \frac{1}{\sqrt{\pi t}}.$$

We've shown that

$$\mathcal{L}\left[\frac{1}{\sqrt{\pi t}}\right] = p^{-1/2},$$

so with the aid of this example and property 6 of the Laplace transform, we can also invert $p^{-3/2}$, $p^{-5/2}$, etc.

The next set of examples show how transforms can help with PDEs as well as ODEs.

(c) Consider the following IBVP for the heat equation in $x \geq 0$, $t \geq 0$:

$$\kappa\ u_{xx} = u_t$$

$$u(x,0) = 0 \text{ initial condition}$$

$$\left.\begin{array}{r} -ku_x(0,t) = Q \\ u \to 0 \text{ as } x \to \infty \end{array}\right\} \text{boundary conditions}$$

Find $u(0,t)$ for $t \geq 0$.

The problem corresponds to a semi-infinite bar, initially at zero temperature, which has a constant rate of heat-flow into it at the origin. Heat will flow in and pass along the bar and in particular the end at the origin will heat up. We are asked for the temperature of this end.

We Laplace transform in t only (we only care about $t \geq 0$):

$$\bar{u}(x,p) = \int_0^\infty u(x,t)e^{-pt}dt.$$

Then, by property 3 of Laplace transform, and using the heat equation:

$$p\bar{u} = \mathcal{L}[u_t] = \mathcal{L}[\kappa u_{xx}]$$

$$= \kappa \int_0^\infty u_{xx} e^{-pt} dt = \kappa \bar{u}_{xx}.$$

This is an ODE in x for $\bar{u}$ which we can solve:

$$\bar{u} = A(p) e^{x\sqrt{p/\kappa}} + B(p) e^{-x\sqrt{p/\kappa}}$$

but $\bar{u} \to 0$ as $x \to \infty$. Therefore, $A = 0$ and for B we note

$$\bar{u}_x(0, p) = \int_0^\infty u_x(0, t) e^{-pt} dt$$

$$= -\frac{Q}{k} \int_0^\infty e^{-pt} dt$$

$$= -\frac{Q}{k} \left[-\frac{1}{p} e^{-pt} \right]_0^\infty$$

$$= -\frac{Q}{kp}$$

$$= -\sqrt{\frac{p}{\kappa}} B(p)$$

so that

$$B = \frac{Q\sqrt{\kappa}}{k} p^{-3/2},$$

whence

$$\bar{u}(x, p) = \frac{Q\sqrt{\kappa}}{k} p^{-3/2} e^{-x\sqrt{p/k}}.$$

Inverting this would give us $u(x, t)$, but we are only asked for $u(0, t)$. For this

$$\bar{u}(0, p) = \frac{Q\sqrt{\kappa}}{k}\ p^{-3/2}.$$

To invert this, go back to example (b):

$$\mathcal{L}\left[\frac{1}{\sqrt{t\pi}}\right] = \frac{1}{\sqrt{p}},$$

so

$$\frac{d}{dp}(p^{-1/2}) = -\frac{1}{2} p^{-3/2}$$

$$= -\mathcal{L}\left[t \times \frac{1}{\sqrt{\pi t}}\right]$$

using property 6 of Laplace transform,

$$= -\mathcal{L}\left[\sqrt{\frac{t}{\pi}}\right]$$

so that $p^{-3/2}$ inverts to $2\sqrt{\frac{t}{\pi}}$, and

$$u(0,t) = \frac{2Q}{k}\sqrt{\frac{\kappa t}{\pi}}.$$

(d) An IVP for the heat equation in an infinite bar:

$$u_{xx} = u_t; \quad -\infty < x < \infty, \quad t \geq 0$$

$$u(x,0) = f(x).$$

This time we Fourier transform in x:

$$\hat{u}(\omega,t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx$$

so

$$\hat{u}_t = \int u_t e^{-i\omega x}dx = \int u_{xx}e^{-i\omega x}dx = -\omega^2\hat{u},$$

an ODE we can solve:

$$\hat{u}(\omega,t) = A(\omega)e^{-\omega^2 t}$$

but

$$\hat{u}(\omega,0) = \hat{f}(\omega) \text{ therefore } A = \hat{f}$$

and

$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-\omega^2 t}.$$

This is a product of Fourier transforms, so corresponds to a convolution:

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t)\ f(y)dy \tag{6.3}$$

where

$$\hat{K}(\omega,t) = e^{-\omega^2 t}.$$

By the inversion formula (6.2)

$$K(x,t) = \frac{1}{2\pi} \lim_{R\to\infty} \int_{-R}^{R} e^{i\omega x} e^{-\omega^2 t} d\omega$$

so with $s = \omega\sqrt{t}$ this is

$$= \frac{1}{2\pi\sqrt{t}} \lim_{R\to\infty} \int_{-R}^{R} e^{-s^2+isx/\sqrt{t}} ds.$$

We can evaluate this integral by the following Lemma.

Lemma: for real a

$$\int_{-\infty}^{\infty} e^{-s^2+2ias} ds = \int_{-\infty}^{\infty} e^{-(s-ia)^2}\, e^{-a^2} ds = \sqrt{\pi} e^{-a^2}$$

Proof

By Cauchy's integral formula

$$\oint_{\Gamma} e^{-(z-ia)^2} dz = 0$$

where Γ is the rectangle with vertices at $(R, R+ia, -R+ia, -R)$ for positive, real R. Now take R to infinity and show that the integrals along the short sides tend to zero, so that

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-(x-ia)^2} dx = \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2} dx = \sqrt{\pi}$$

With the aid of this Lemma, we have

$$K(x,t) = \frac{1}{2\pi\sqrt{t}} \cdot \sqrt{\pi} e^{-x^2/4t} = \frac{e^{-x^2/4t}}{2\sqrt{\pi t}}, \tag{6.4}$$

and then

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x-y)^2} f(y) dy. \tag{6.5}$$

It's worth making a few points about this calculation:

- note the resemblance of (6.3) to Green's function expressions from section 2, like (2.12) or (2.16);

- note that $K(x,t)$ in (6.4) has appeared before (apart from a constant factor), in (5.11); it is a solution of the heat equation which, at $t = 0$, is zero everywhere except at the origin, where it is infinite; from its resemblance to the normal distribution, we know that the integral over x of K is one at all times - intuitively it is like a unit point source of heat at time zero spreading out;
- we can think of (6.5) as expressing $u(x,t)$ as a *superposition* of point sources distributed according to the initial temperature given by f and spreading out;
- as we saw in the discussion of well-posed-ness, the solution is defined for positive t only.

(e) The Poisson kernel

Something similar to the last example is possible for the BVP for the Laplace equation. Consider the problem:

$$u_{xx} + u_{yy} = 0 \quad -\infty < x < \infty, \quad y > 0$$

$$u(x,0) = f(y)$$

$$u \to 0 \text{ as } y \to \infty.$$

So this is the BVP with data on the real axis. We Fourier transform with respect to x:

$$\hat{u}(\omega, y) = \int_{-\infty}^{\infty} e^{-i\omega x} u(x,y) dx$$

and then use the Laplace equation and property 2 of the Fourier transformation to find

$$\hat{u}_{yy} = \omega^2 \hat{u},$$

with $\hat{u}(\omega, 0) = \hat{f}(\omega)$, and $\hat{u} \to 0$ as $y \to \infty$. The solution of this ODE for $\hat{u}$ needs care. I claim that

$$\hat{u}(\omega, y) = \hat{f}(\omega) e^{-|\omega| y}.$$

This is a product, so the answer will be a convolution. For the inversion of the second factor, we consider the integral

$$\frac{1}{2\pi} \int_{-R}^{R} e^{i\omega x - |\omega| y} d\omega$$

$$= \frac{1}{2\pi}\int_{-R}^{0} e^{\omega(ix+y)}d\omega + \frac{1}{2\pi}\int_{0}^{R} e^{\omega(ix-y)}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{\omega(ix+y)}}{ix+y}\right]_{-R}^{0} + \frac{1}{2\pi}\left[\frac{e^{\omega(ix-y)}}{ix-y}\right]_{0}^{R}$$

and for large R:

$$\to \frac{1}{2\pi}\left(\frac{1}{ix+y} - \frac{1}{ix-y}\right) = \frac{y}{\pi(x^2+y^2)} := K(x,y). \tag{6.6}$$

Therefore

$$u(x,y) = \int_{-\infty}^{\infty} K(x-s,\ y)f(s)ds \tag{6.7}$$

$$= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2+y^2}ds. \tag{6.8}$$

The function K is called the *Poisson kernel.*

The remarks after the previous example carry over here: (6.7) is like a Green's function formula; K in (6.6) is a solution of the Laplace equation singular at one point; (6.8) expresses u as a superposition of these elementary solutions along the boundary.

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Figure captions

Figure 1.1: the rectangle R.

Figure 1.2: successive iterates graphed in R.

Figure 1.3: the region of integration for the interchange trick (shaded).

Figure 3.1: the phase portrait for the harmonic oscillator; to put the arrows on the trajectories, notice that $\dot{x} > 0$ if $y > 0$.

Figure 3.2: an unstable node.

Figure 3.3: a stable node.

Figure 3.4: a saddle.